1. **Group theory:** Consider a group of order 3 with multiplication table of the form

\[
\begin{array}{c|ccc}
 & e & a & b \\
\hline
e & e & a & b \\
a & a & b & ? \\
b & b & ? & ? \\
\end{array}
\]

a. What do the values of the ?'s have to be. Why?
b. Show \( D(e) = 1, D(a) = e^{2\pi i/3}, D(b) = e^{4\pi i/3} \) is a representation of this group.
c. Is this representation reducible?
d. What are the characters of this representation?

2. **Lie Groups/Lie Algebras:** The Lie Algebra for \( SU(2) \) has the form

\[
[L_i, L_j] = i \sum_{j=1}^{3} \epsilon_{ijk} L_k
\]

a. Write down elements of the adjoint representation of this Lie algebra.
b. Calculate \( \text{Tr}(L_i L_j) \).
c. What is the Cartan sub algebra of this representation?

3. **Orthogonal Polynomials:** The classical orthogonal polynomials are defined by

\[
C_n(x) = \frac{1}{w(x)} \frac{d^n}{dx^n} (w(x) s^n(x))
\]

where \( w(x) > 0, s(x) \) is a polynomial of degree less than or equal to two, \( C_1(x) \) is a polynomial of degree 1, and \( w(a)s(a) = w(b)s(b) = 0 \). Consider the case where \( s(x) = 1/2 \) and \( C(x) = -x \).

a. Find an equation for the weight function \( w(x) \). Find the solution.
b. Find the points \( a \) and \( b \) satisfying \( w(a)s(a) = w(b)s(b) = 0 \).
c. Show that \( C_n(x) \) defined this way are polynomials in \( x \).

4. **Distributions/Fourier Transforms** The delta function is the continuous linear functional acting on the space of Schwartz functions defined by

\[
\int \delta(x - y) f(y) dy = f(x)
\]

where \( f(x) \) is a Schwartz function. Let \( f(x) = x^2 e^{-ax^2} \) with \( a > 0 \). Calculate
1a) since the rows and columns of the multiplication table have to have different elements

\[
\begin{array}{ccc}
  e & a & b \\
  e & e & a & b \\
  a & a & b & e \\
  b & b & e & a \\
\end{array}
\]

1b) \( D(a) D(a) = D(b) \)
\( D(a) D(b) = D(b) D(a) = D(c) \)

This shows that this representation has the same multiplication table as
\( D(a) D(b) = D(a b) \)

1c) no - it is one dimensional

1d) \( e^{2 \pi i / 3} e^{4 \pi i / 3} (\text{traces of } D(a)) \)

2c)
\[
L_1 = -i E_{ijk} = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}
\]

\[
L_2 = -i E_{2jk} = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]

\[
L_3 = -i E_{3jk} = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
\[ L_1^2 L_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Tr} = 0 \]

\[ L_2^2 L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Tr} = 0 \]

\[ L_3^2 L_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Tr} = 0 \]

\[ \text{Tr} (L_1^2 L_1) = \text{Tr} (L_2^2 L_3) = \text{Tr} (L_3^2 L_1) = 2 \]

3) Take any one generator \( \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

3a) \[ -x = \frac{1}{w} \frac{d \omega}{dx}, \quad \frac{1}{2} \]

\[ \frac{d \omega}{\omega} = -2x \]

\[ \int \frac{d \omega}{\omega} = -x^{\frac{3}{2}} = \ln \omega \]

\[ \omega = e^{-x^\frac{3}{2}} \]

3b) This vanishes at \( a = -\infty \) \( b = +\infty \)

3c) \[ C(x) = \frac{1}{2} e^{x^2} \left( \frac{d}{dx} \right)^n \left( \frac{1}{2} e^{-x^2} \right) \]

Since \( \frac{d}{dx^n} \left( e^{-x^2} \right) = e^{-x^2} \) polynomials of degree \( n \), \( C(x) = \) polynomial of degree \( n \).
\[ 4.9) \quad - \frac{df}{dx}(x) = -(2x - 2ax^3)e^{-ax^2} \]

\[ 4.10) \quad \frac{d^2f}{dx^2}(x) = \left(2 - 6ax^2 - 4ax + 4a^2x^4\right)e^{-ax^2} = 2(1 - 5ax^2 + 2a^2x^4)e^{-ax^2} \]

\[ 4.11) \quad \frac{1}{|b|} f(x) = \frac{1}{|b|} x^2 e^{-ax^2} \]