Final exam review

• Group theory/group representations

\[
g_1 g_s \in G \quad e \in G \quad g^{-1} \in G \quad (g_1 g_2) g_3 = g_1 (g_2 g_3)
\]

Order = number of elements of \( G \)

Multiplication table - no repeated elements in each row or column

Abelian \( g_1 g_2 = g_2 g_1 \), non-abelian otherwise

Permutation groups, cyclic groups,

• Representations \( D(g_2) D(g_1) = D(g_2 g_1) \), \( D(g) \) linear operator on a vector space.

• Regular representations \( D(g_1)|g_2\rangle = |g_1 g_2\rangle \)

Equivalent representations \( D'(g) = S G(g) S^{-1} \)

• Irreducible representations (no non-trivial invariant subspace)

\[
D(g) P = P D(g) P \quad \forall g \in G \quad P = 0, I
\]

• Completely reducible representations (equivalent to block diagonal representation with irreducible blocks)

• Theorem: Every representation of a finite group is equivalent to a unitary representation. - counter-example for infinite group

• Corollary: Every reducible representation of a finite group is completely reducible.

• Subgroups \( H \subset G, H \) group

Right cosets \( g' \in [g]\) if \( g = h g' \ h \in H \)

Left cosets \( g' \in [g]\) if \( g = g' h \ h \in H \)

Normal subgroup: left cosets = right cosets

Cosets of normal subgroup form group = quotient group \( G/H \)

Center for group \( c \in C \) if \( c g = g c \ \forall g \in G \)

• Conjugacy classes \( c' \in [c] \) if \( c' = g c g^{-1} \)

Isomorphism \( G \rightarrow G' \) invertible, preserving all group operations

Homomorphism \( G \rightarrow G' \) preserving all group operations
Automorphism $G \to G$ isomorphism from $G$ to $G$

Inner automorphisms, $g' = g_0gg_0^{-1}$, outer automorphisms = automorphism that his not inner

- Schur’s lemma
  $D_1(g)$, $D_2(g)$ irreducible, $D_1(g)A = AD_2(g)$ then $A = 0$ or $D_1(g)$ equivalent to $D_2(g)$; $D_1(g)A = AD_1(g)$ then $A = \text{const} \times I$

Implications $[O, D(g)] \forall g \in G$

$\langle n, j, m|O|n', j', m' \rangle = \delta_{jj'}\delta_{mm'}O_{nn'}$

- Orthogonality theorem
  $$\sum_{g \in G} D^*_{m'm'}(g)D_{m\bar{m}'}(g) = \frac{N}{n_j} \delta_{jj'}\delta_{m\bar{m}}\delta_{m'\bar{m}'}$$

- Orthonormal basis for functions of $G$
  $$\sqrt{\frac{n_j}{N}} D^j_{m\bar{m}}(g) := \langle g|j, m, \bar{m}\rangle$$

$$\langle j, m\bar{m}|F \rangle = \sum_{g \in G} D^j_{m\bar{m}}(g) \langle g|F \rangle$$

- Characters $\chi_D(g) = \text{Tr}(D(g))$
  this is the same for all equivalent representations
  $$\sum_{g \in G} \chi^j_{m'}(g)\chi^{j'}(g) = N\delta_{jj'}$$

- Characters for different irreducible representations are orthogonal.
- Characters are constant on conjugacy classes
  $$\chi_D(g) = \chi_D(g') = \chi_D([g])$$

$$\sum_{[g]} \chi^j([g])\chi^{j'}([g]) = \frac{N}{k_g} \delta_{jj'}$$

- 1 to 1 correspondence between irreducible representations in the regular representation and conjugacy classes.
- Cayley’s theorem - every finite group is a subgroup of the permutation group.
- Lie Groups $g = g(\alpha)$, $\alpha$ continuous parameters.
Representations \( D(g(\alpha)) \), choice of origin \( D(g(0)) = I \).

- Generators

\[
X_j := -i \frac{\partial}{\partial \alpha_j} D(g(\alpha)|_{\alpha = 0})
\]

Exponential parameterization

\[
U(\lambda) = e^{i\lambda \cdot X} = \lim_{N \to \infty} \prod (1 + i \frac{\lambda \cdot X}{N})^N
\]

\[e^{iA} e^{iB} \neq e^{iB} e^{iA}\]

- Expand in neighborhood of identity, equate coefficients of second order mixed terms

\[ [X_i, X_j] = \sum_k i f_{ijk} X_k \text{ Lie Algebra} \]

- \( f_{ijk} = -f_{jik} \) structure constants.

\[ [T_i]_{jk} - i f_{ijk} \text{ adjoint representation of Lie algebra} \]

Structure constants depend of parameterization; using freedom to reparameterize

\[ [T_i, T_j] = i \sum_k f_{ijk} T_k \]

- with \( f_{ijk} \) completely antisymmetric.

\[ Tr(T_i T_j) = k_i \delta_{ij} \]

Compact Lie groups: all \( k_i > 0; \)

Invariant sub algebra

\[ [Y_i, T_j] = i \sum k f_{ijk} Y_k \]

- Exponential of invariant sub-algebra gives normal subgroup of \( D[G] \)

- Simple groups - no non-trivial invariant subalgebras

- Semi-simple groups - no non-trivial abelian subalgebras

- SU(2)

- Cartan sub algebra = maximal set of linearly independent commuting generators. These can be simultaneously diagonalized.

Rank of Lie algebra = dimension of Cartan sub algebra.
• Adjoint representation

\[ X_i |X_j\rangle = [X_i, X_j] \]

\( X_j \) in Cartan sub algebra

\[ X_i |E_j\rangle = \lambda_j |E_j\rangle \rightarrow [X_i, E_j] = \lambda_j E_j \quad [X_i, E_j^\dagger] = -\lambda_j E_j^\dagger \]

Give analog of raising and lowering operators in adjoint representation.

weights - eigenvalues of operators in Cartan sub algebra

• infinite dimensional vector spaces

\[ \langle f | g \rangle = \int_a^b w(x) f^*(x) g(x) dx \quad w(x) > 0 \]

• Inner product space - not complete if functions are continuous

• Operator norms, uniform, strong and weak convergence.

• Cauchy sequences, completeness, basis

• Lebesgue integral

\( \sigma \)-algebra: \( X, \{ \}, \) countable unions, complements

Borel sets: smallest \( \sigma \)-algebra containing open sets

Lebesgue measure \( \mu([a, b]) = b - a \), measure of countable unions of disjoint sets = sum of measures.

Measurable functions - inverse image of open sets are measurable, extends class of continuous functions (inverse image of open sets are open).

Lebesgue integral \( \int \sum a_n \chi(A_n) = \sum_n a_n \mu(A_n) \) where \( \chi(A_n) = 1 \) on measurable set \( A_n \) and 0 on the complement.

• Riesz-Fischer - \( L^p(R) \) complete with Lebesgue measure. Functions that differ on sets of measure 0 are identified, Cauchy sequences are measurable and define new vectors

• Orthonormal bases - complete and independent

• Bessel’s inequality and Parseval’s relation:

\[ \sum |\langle e_n | f \rangle|^2 \leq \langle f | f \rangle \]

\[ \sum |\langle e_n | f \rangle|^2 = \langle f | f \rangle \quad |e_n\rangle \quad \text{basis} \]
• Weierstrass Theorem - polynomials converge uniformly to continuous functions on finite intervals. The Riesz-Fischer extends this to square integrable functions if uniform convergence is replaced by convergence in the mean. Orthogonal polynomials with any weight can be constructed using the Gram Schmidt method.

• Classical orthogonal polynomials

\[ C_n(x) = \frac{1}{w(x)} \frac{d^n}{dx^n}(w(s^n(x))) \]

degree \( s(x) \leq 2 \), degree \( C_1(x) = 1 \), \( w(x) > 0 \) and \( w(a)s(a) = w(b)s(b) = 0 \)

• \( C_n(x) \) orthogonal polynomials on \([a,b]\) with weight \( w(x) \), \( s(x) = \text{const} \) gives Hermite polynomials, \( s(x) = ax + bx \) gives associated Laguerre polynomials, and \( s(x) = a + bx + cx^2 \) gives Jacobi polynomials, which include Legendre, Gegenbauer and both kinds of Chebyshev polynomials

• All orthogonal polynomials satisfy 3 term recursion relations.

• The classical orthogonal polynomials are solutions to second order ordinary differential equations. (the equations also have non-polynomial solutions).

• Gaussian integration. \( x_n \) zeroes of \( P_N(x) \) and

\[ w_n = \int w(x) \prod_{m \neq n} \frac{x - x_m}{x_n - x_m} \]

\[ \int w(x)f(x)dx \approx \sum_{n=1}^{N} f(x_n)w_n \]

exact for \( f(x) \) a polynomial of degree \( < 2N \)

• Fourier series

\[ \frac{1}{\sqrt{2\pi}} e^{inx} f(x) = \sum_{n=-\infty}^{\infty} f_n \frac{1}{\sqrt{2\pi}} e^{inx} \quad f_n = \int f(x) \frac{1}{\sqrt{2\pi}} e^{-inx} \]

orthonormal basis for periodic continuous functions on \([-\pi, \pi]\) (uniform convergence follow from Weierstrass)

• Convergence in the mean for non-periodic continuous functions.

• Can be expressed in terms of \( \sin(nx) \) and \( \cos(nx) \)

• Tempered distributions - Schwartz functions

\[ \langle f|(-\frac{d^2}{dx^2} + x^4 + 1)^m|f \rangle < \infty \]
- Schwartz function, Schwartz distributions (linear functionals on Schwartz functions)

\[ f \in \mathcal{S} \quad \sum |\langle f | n \rangle|^2 (2(n+1))^m < \infty \quad \forall m < \infty \]

\[ f \in \mathcal{S}' \quad |\langle f | n \rangle|^2 \leq (2(n+1))^m \text{ for some } m \]

- derivatives, heaviside function, fourier transforms, delta functions

\[ \langle f'| g \rangle := -\langle f | g' \rangle \quad f \in \mathcal{S}' \quad g \in \mathcal{S} \]

- Fourier transform

Fourier transform is unitary

Fourier transform maps Schwartz functions to Schwartz functions

Fourier transform maps Schwartz distributions to Schwartz distributions

- linear operators - unbounded, bounded, compact

- wavelets - fractal valued orthonormal basis

\[ s(x) = \sum \sqrt{2} h_n s(2x-l) \quad \int s(x) = 1 \]

\[ w(x) = \sum \sqrt{2} (-)^n h_{2K-n} s(2x-l) \quad \int w(x) = 0 \]

\[ s_n^k(x) = D^{k+T} s(x) \quad w_n^k(x) = D^{k+T} w(x) \]

\( \{ s_n^k \} \cup \{ w_n^{k+1} \} \quad k \text{ fixed, } l \geq 0 \)

orthonormal basis.

- Compact operators (definition)

These are operators \( O \) that can be uniformly approximated by finite dimensional matrices \( F_N \):

\[ \| O - F_N \| < \epsilon \]

- Compact operators (general representation)

\[ O = \sum_{n=1}^{\infty} |\phi_n \rangle \lambda_n \langle \psi_n | \quad \lambda_n \to 0 \quad \langle \phi_n | \phi_m \rangle = \langle \psi_n | \psi_m \rangle = \delta_{mn} \]

- Compact operators (test - Hilbert Schmidt operators - sufficient for compactness)

\[ \text{Tr}(O^\dagger O) < \infty \]
• Fredholm integral equations ($O$ compact)

$$|\phi\rangle = |\phi_0\rangle + O|\phi\rangle \quad |\phi\rangle = (1 - F_N)^{-1}|\phi_0\rangle + \left(1 - F_N\right)^{-1} (O - F_N)|\phi\rangle$$

finite matrix inversion small

• Analytic Fredholm Theorem

For $O(z)$ compact and analytic in a region $R$, then for $z_0 \in R$ either $(I - O(z))^{-1}$ does not exist for $z = z_0$ or it is a bounded analytic function in a neighborhood of $z_0$.

• Volterra integral equations

$$|\phi(t)\rangle = |\phi(0)\rangle + \int_0^t O(t')|\phi(t')\rangle dt'$$

Here the $t$-dependence is in the integration bounds; these equations arise in time dependent perturbations theory and quantum field theory.

• Dyson series

$$|\phi(t)\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^t dt_1 \cdots dx_n T(O(t_n) \cdots O(t_1))|\phi(0)\rangle$$

uses a trick, replace the product by a time ordered product, to write the solution in exponential form where it is easy to show the iterative solution converges for finite $t$ and bounded $O(t)$. Dyson used this to generate Feynman diagrams.

• Differential equations - general form

$$F(u, \frac{du}{dx}, \cdots \frac{d^n}{dx^n} x) = 0$$

• Reduction of $n$-th order equation to a system of $n$ first order equations.

Solve for

$$\frac{d^n u}{dx^n} = G(u, \frac{du}{dx}, \cdots \frac{d^{n-1}}{dx^{n-1}}, x)$$

This can be converted to a system of $N$ coupled first order equations:

$$\frac{du_1}{dx} = u_2, \cdots \frac{du_{n-2}}{dx} = u_{n-1}, \frac{du_{n-1}}{dx} = G(u, u_1, \cdots u_{n-1}, x)$$

• Local existence - Cauchy Lipschitz

The differential equation can be solved locally if $G$ satisfies a Lipschitz condition - differentiability is sufficient for the Lipschitz condition to hold. The solution is generated by converting this to an integral equation and iterating.
• Homogeneous boundary conditions (for N-th order)
  \[ \sum_{n=0}^{N-1} a_{mn} \frac{d^nu}{dx^n}(x_m) = 0 \quad 0 \leq m < N \]

• Inhomogeneous boundary conditions
  \[ \sum_{n=0}^{N-1} a_{mn} \frac{d^nu}{dx^n}(x_m) = b_m \quad 0 \leq m < N \]

• Conditions for independent solutions
  Determinant of independent solutions and their first \( N - 1 \) derivatives should be non-zero

• Wronskian
  \[ W = \det(\frac{d^n u_m}{dx^n}) \]
  for \( N = 2 \)
  \[ W = u_1' u_2 - u_1 u_2' \]

• Second order linear differential equations
  \[ a(x)u''(x) + b(x)u'(x) + c(x)u(x) = f(x) \]
  The equation is homogeneous of \( f(x) = 0 \), otherwise it is inhomogeneous.
  The homogeneous equation has 2 independent solutions.
  Linear combinations of these can be added to solutions of the inhomogeneous
  equation to satisfy the boundary conditions.
  boundary conditions on \([a, b]\)
  \[ c_{1i}u_i(a) + c_{2i}u_i(b) + c_{3i}u'_i(a) + c_{4i}u'_i(b) = d_i \quad i = 1, 2 \]

• Finding the second solution of the homogeneous equation given the first solution.
  Use \( u_2(z) = u_1(z)h(z) \), solve for \( h(z) \) - first order equations.
  trick is to let
  \[ \frac{p'}{p} = \frac{b}{a} \]

• Finding a solution of the inhomogeneous equation given a first solution of the homogeneous equations.
  Use \( u_2(z) = u_1(z)h(z) \), solve for \( h(z) \) - first order equations.
Trick

\[
\frac{r'}{r} = \frac{b}{a} + 2 \frac{u_1'}{u_1}
\]

- Lagrange's identity, Generalized Green’s identity, adjoint operators and boundary conditions

\[
w(x)(v^*(x)L_x u(x) - u(x)L_x^\dagger(x)v^*(x)) = \frac{dF(u,v^*)}{dx}
\]

The adjoint operator, \(L_x^\dagger\), defined so the RHS is a total derivative

\[
\int_a^b w(x)(v^*(x)L_x u(x) - u(x)L_x^\dagger(x)v^*(x))dx = F(u,v^*)(b) - F(u,v^*)(a)
\]

This is called the generalized Green’s identity

Given boundary conditions, adjoint boundary conditions are defined to make the right side of the generalized Green’s identity vanish. When this happens it is called Green’s identity.

- Self-adjoint operators.
  - \(L\) is self adjoint if \(L_x = L_x^\dagger\) and boundary conditions and adjoint boundary conditions are the same.
  
  Self adjoint operator have real eigenvalues and eigenvectors that are orthogonal if the eigenvalues are different - just like matrices.

- All second-order differential equations with real coefficients can be made self adjoint given a appropriate weight. For \(p\) defined above:

\[
w = \frac{p}{a}
\]

- Greens functions are right inverses to the the differential operator. They are solutions to

\[
L_x G(x, y) = \delta(x - y) \quad L_x^\dagger g(x, y) = \delta(x - y)
\]

satisfying the boundary conditions of the problem.

- Existence and uniqueness.
  
  Exists and is unique when there are no solutions to \(L_x u = 0\) satisfying homogeneous boundary conditions (necessary and sufficient). This can be seen from the analytic Fredholm theorem

- Properties

\[
G(x, y) = g^*(y, x)
\]

For \(L = L^\dagger\)

\[
G(x, y) = g(x, y) = G^*(y, x)
\]
• Using Green’s functions to solve inhomogeneous second order differential equations.

\[ Lu = f \quad u = Gf \quad L^1 g = I \]

• Calculation’s of Green’s functions.

Linear combination of solutions of homogeneous equation when \( x \neq y \), satisfying BC at \( x = a \) and \( x = b \), continuous at \( x = y \), derivative is discontinuous at \( x = y \) increases by 1.

• Using Green’s functions to convert differential equations to integral equations.

\[ L_\nu u = -\nu u \quad u = \nu Gu \]

• Generalized Green’s functions (conditions).

\( f \) must be in the range of \( G \), in order to get a unique solution \( u \) must be orthogonal to the null space of \( G \). Use bc on \( G \) and inhomogeneous BC on \( u \) to eliminate the rhs of the generalized Green’s identity. The generalized Green’s function is the Moore Penrose generalized inverse of \( L_z \) when the inverse does not exist.

• Equations with inhomogeneous boundary conditions.

Assume \( L_x G = I \) and \( G \) satisfies the homogeneous boundary conditions, but do not assume that \( u \) satisfies the homogeneous boundary conditions. Use the generalized Green’s identity with Green’s function satisfying homogeneous boundary conditions with the inhomogeneous boundary conditions for \( u \).

• Strum Liouville operators;

These are self adjoint differential operators. Greens function is a self-adjoint compact operator with non-zero real eigenvalues that approach 0. This can be seen by showing that \( G \) is a Hilbert-Schmidt operator. These operators have a complete set of orthogonal eigenvetors.

• Series representation of the Green’s function.

This is simply the eigenfunction expansion of the Green’s function.

The inverse of \( L_x - z \) for \( L_x = L^1_x \) is \( G_z(x, y) \)

\[ G_z(x, y) \sum \frac{\langle x|n\rangle\langle n|y\rangle}{\lambda_n - z} \quad L_x|n\rangle = \lambda_n|n\rangle \]

• Classification of second order equations -

\( z_0 \in \mathbb{R} \) is an ordinary point if \( b(z)/a(z) \) and \( c(z)/a(z) \) are analytic at \( z = z_0 \).
$z_0 \in R$ is a regular singular point if $z_0$ not an ordinary point and $(z - z_0)b(z)/a(z)$ and $(z - z_0)^2c(z)/a(z)$ are analytic at $z_0$.

$z_0 \in R$ is irregular singular point if $p(z)$ and $q(z)$ are singular at $z = z_0$ is not a regular singular point.

- Series solution of differential equation (ordinary points)
  Both solutions are analytic. Determined by $u(z_0)$ and $u'(z_0)$
  - Convergence
    The differential equation gives a recurrence relations for the coefficients of the power series. We showed that the series converges uniformly in a neighborhood of $z_0$.
  - Series solution of differential equation (regular singular points - indicial equations)
    Solution has the form $(z - z_0)^r F(z)$ where $F$ is analytic. The numbers $r$ are roots of the indicial equation that depends on the limit of $(z - z_0)b/a$ and $(z - z_0)^2c/a$ as $z \to z_0$.
    When the roots differ by an integer amount - the series method does not work for the second solution - but we can use the method that generates the second solution from the first in that case.
  - Convergence of the analytic part was demonstrated.
  - Integral representations
    \[
    u(z) = \int K(z, t)v(t)dt \quad L_z u(z) = 0
    \]
    \[
    L_x K(x, t) = M_t K(x, t) \quad M_t^1 v(t) = 0
    \]
    Extends domain of analyticity
  - Fuchsian equations with three regular points
    9 parameter set of equations - 3 singular points and two roots of the indicial equation about each singular point.
    Using homographic transformations and some other transformation that preserve the form of the equations, the 9 parameteres can be expressed in term of 3. One root is at 0 and one root of the indicial equation at 0 is 0, so there is one analytic solution in a neighborhood of the origin. That is the hypergeometric function. It is normalized so it is 1 at $z = 0$
  - Hypergeometric equation/function
This is the analytic solution of the hypergeometric equation in the neighborhood of the origin. It has an analytic series solution. The second solution at the origin can be expressed in terms of the analytic solutions with different parameters.

- Solutions with different arguments.

The series solution can be used to relate Hypergeometric functions with different arguments.

- The solutions of the equation in the neighborhood of each singular point are similarly related to the hypergeometric function with different sets of arguments and $z \to (1 - z)$ and $z \to 1/z$.

- Integral representations and series solution.

An integral representation is constructed using a kernel of the form

$$K(z, t) = (z - t)^\lambda$$

The series solution is related to the integral representation.

- Relation to special functions

The Hypergeometric function is related to the solutions of the Jacobi and Gegenbauer differential equations. Under some conditions they become polynomials.

$$\left(1 - z\right)^2 \frac{d^2 P_\lambda^{(\alpha, \beta)}(z)}{dz^2} + \left(\beta - \alpha + \beta + 2\right)z \frac{dP_\lambda^{(\alpha, \beta)}(z)}{dz} + \lambda(\lambda + \alpha + \beta + 1)P_\lambda^{(\alpha, \beta)}(z) = 0$$

and the Gegenbauer equation

$$\left(1 - z\right)^2 \frac{d^2 G_\lambda^{(\mu)}(z)}{dz^2} + (2\mu + 1)z \frac{dG_\lambda^{(\mu)}(z)}{dz} + \lambda(\lambda + 2\mu)G_\lambda^{(\mu)}(z) = 0.$$
• Confluent hypergeometric function

These are a limit of the Hypergeometric equations where one of the roots is moved to infinity.

\[ z \frac{d^2 u(z)}{dz^2} + (c - z) \frac{du(z)}{dz} - au(z) = 0. \]

\[ \Phi(a, c; z) = \lim_{b \to \infty} F(a, b, c; \frac{z}{b^2}). \]

• Relation to special functions

Solutions are related to laguerre functions, hermite functions, error functions and Bessel functions.

\[ \Psi(a, c; z) = \Gamma(c - 1) \frac{\Phi(a, c; z)}{\Gamma(a - c + 1)} z^{1-c} \Phi(a - c + 1, 2 - c, z) \]

Parabolic cylinder functions

\[ D_\nu(z) = 2^{\nu/2} e^{-z^2/4} \Psi(-\nu, 1; \frac{1}{2}; \frac{z^2}{2}) \]

Hermite polynomials

\[ H_n(z) = 2^n \Psi(-\frac{n}{2}, \frac{1}{2}; \frac{z^2}{2}) = 2^{\nu} e^{i\nu} D_n(z) \]

Associated Laguerre polynomials

\[ L_n^\mu(z) = \frac{\Gamma(n + \mu + 1)}{\Gamma(n + 1) \Gamma(\mu + 1)} \Phi(-n, \mu + 1, z) \]

Error Function

\[ Erf(z) = z \Phi\left(\frac{1}{2}, \frac{3}{2}; -z^2\right) \]

• Bessel functions

\[ J_\nu(z) := \frac{1}{\Gamma(\nu + 1)} \left(\frac{z}{2}\right)^\nu e^{-iz} \Phi(\nu + \frac{1}{2}, 2\nu + 1; 2iz) \]

\[ J_{\pm\nu}(z) \text{ independent when } \nu \neq n \]

\[ Y_n(z) := \lim_{\nu \to n} Y_\nu(z) = \lim_{\nu \to n} \frac{1}{\pi} \left\{ \frac{\partial J_\nu(z)}{\partial \nu} - (-)^n \frac{\partial J_{-\nu}(z)}{\partial \nu} \right\} \]

• Series expansion of Bessel functions.

\[ J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \nu + n)} \left(\frac{z}{2}\right)^{2n+\nu}. \]
• Integral representations

\[ J_\nu(z) = a \int_C K(z, t) v(t) dt = \frac{1}{2\pi i} \left( \frac{z}{2} \right)^\nu \int_C e^{t-z^2/4} t^{-\nu-1} dt. \]

\[ J_\nu(z) = \frac{1}{2\pi i} \int_C e^{\frac{z}{2}(u-1/u)} u^{-\nu-1} du. \]

• Recurrence relations

\[ \frac{dJ_\nu(z)}{dz} = -\frac{\nu}{z} J_\nu(z) + J_{\nu-1}(z) \]

\[ \frac{dJ_\nu(z)}{dz} = \frac{\nu}{z} J_\nu(z) - J_{\nu+1}. \]

\[ \frac{2\nu}{z} J_\nu(z) = J_{\nu+1}(z) + J_{\nu-1}(z). \]

• Bessel functions of imaginary argument

\[ I_\nu(z) = e^{-\frac{i\pi \nu}{2}} J_\nu(iz) = \]

\[ K_\nu(z) := \frac{\pi}{2\sin(\nu\pi)} [I_\nu(z) - I_{-\nu}(z)] \]

• Generating functions

\[ e^{\frac{2}{z}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(z) \]

• Fourier Bessel series

\[ f(x) = \sum f_n N_n J_\nu(k_n x) \quad f_n = \int_0^1 N_n J_\nu(k_n x) f(x) x dx \quad N_n = \sqrt{\frac{2}{J_{\nu+1}(k_m)}}. \]

• Cauchy Kovaleska Theorem - existence of local solutions of partial differential equations. Boundary conditions on flat surface.

• General surfaces - classification of equations (elliptic, hyperbolic, parabolic)

• Characteristic surfaces - Cauchy conditions wrong boundary conditions.

• Boundary conditions for different types of equations.

• Multidimensional Fourier transforms

• Differential forms

• Green’s functions for partial differential equations.

• Singular part of Green’s function.
• Boundary conditions
• Images
• Separation of variables.
• Laplaces equation and spherical harmonics