1. Calculate the Fourier transforms of

\[ f(x) = \frac{1}{(x^2 + a^2)(x^2 + b^2)} \]

\[ f(x) = \begin{cases} 
1 & |x| < 1 \\
0 & |x| \geq 1 
\end{cases} \]

\[ \theta(x) = \begin{cases} 
1 & x > 0 \\
0 & x \leq 0 
\end{cases} \]

2.) Fourier Series: In class we developed the Fourier series for periodic functions on the interval \([-\pi, \pi]\). We constructed the orthonormal basis functions

\[ \langle \theta | e^+_0 \rangle = \frac{1}{\sqrt{2\pi}} \]

\[ \langle \theta | e^+_n \rangle = \frac{1}{\sqrt{\pi}} \cos(nx) \quad n > 1 \]

\[ \langle \theta | e^-_n \rangle = \frac{1}{\sqrt{\pi}} \sin(nx) \quad n > 1 \]

Find the corresponding orthonormal basis functions for periodic functions on the interval \([0, L]\)?

3.) Let \( f(\theta) \) be 1 for \( 0 < \theta < \pi \) and -1 for \( -\pi < \theta \leq 0 \). Find the coefficients \( f_n \) in the Fourier series

\[ f(\theta) = \sum_n f_n \langle \theta | n \rangle \]

4.) Consider the set of rational numbers between zero and one. Show that it is possible to place each rational in the interior of an open interval (i.e. \( a < (m/n) < b \)) in such a way that if we discard all rationals and all of the open sets containing these rationals that what remains in the interval \([0, 1]\) has a measure as close to 1 as desired. [This exercise was a critical element in Kolomogorov, Arnold, and Moser’s solution to the classical three-body problem which asks whether the solar system is stable]

5.) Let

\[ g(x, t) = \frac{1}{\sqrt{1 - 2xt + t^2}} \]

Show that

\[ g(x, t) = \sum_{n=0}^{\infty} t^n P_n(x) \]
where $P_n(x)$ is the $n-th$ degree Legendre polynomial.

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) -nP_{n-1}(x)$$

with $P_0(x) = 1, P_1(x) = x$. It satisfies

$$(1 - x^2)P''_n - 2xP'_n + n(n + 1)P_n = 0$$

The function $g(x,t)$ is called a generating function.

6.) The Schrödinger equation for a harmonic oscillator has the form

$$(-\frac{d^2}{dx^2} + x^2)F_n(x) = (2n + 1)F_n(x).$$

Assume that

$$F_n(x) = C_n(x)e^{-\frac{1}{2}x^2}.$$

Show that $C_n$ satisfies the same differential equation as one of the classical orthogonal polynomials (see page 211-215 in the text).