1.) Let $L_x = \frac{d^2}{dx^2}$ on the interval $[a, b]$.

Find the Green function for this operator corresponding to Dirichlet boundary conditions ($u(a) = u(b) = 0$).

The independent solutions to the homogenous equation are 1 and $x$. This means the the Green’s function satisfying the boundary conditions at $a$ and $b$ have the form

$$G(x, y) = \begin{cases} c(x - a) & x < y \\ d(b - x) & x > y \end{cases}.$$ 

Continuity at $y$ requires

$$c(y - a) = d(b - y)$$

or

$$\frac{c}{d} = \frac{b - y}{y - a}.$$ 

The other requirement is that the derivative at $x = a$ increase by 1 (i.e. the integral over the delta function) which means:

$$-d - c = 1.$$ 

Solving for $c$ and $d$ gives

$$c = -\frac{b - y}{b - a} \quad d = \frac{a - y}{b - a},$$

which gives

$$G(x, y) = \begin{cases} -\frac{(b - y)(x - a)}{b - a} & x < y \\ -\frac{(y - a)(b - x)}{b - a} & x > y \end{cases}.$$ 

This function satisfies all of the required properties.

Find the Green function for this operator corresponding to Neumann boundary conditions ($u'(a) = u'(b) = 0$).

For the Neumann boundary conditions $x = 1$ is a non-zero solution of the homogeneous equation satisfying the boundary condition so the Green’s function does not exist.

2. Consider the linear differential operator

$$L_x = e^x \frac{d^2}{dx^2} - x^2 \frac{d}{dx} + 1$$

Find the adjoint operator on the interval $[0, 1]$.
Find the boundary conditions that are adjoint to Dirichlet boundary conditions \((u(0) = u(1) = 0)\).

Consider
\[
\int_0^1 dx v^*(x) \left( e^x u''(x) - x^2 u'(x) + u(x) \right) =
\int_0^1 dx \left( -(e^x v^*(x))' u'(x) + (x^2 v^*(x))' u(x) + v^*(x) u(x) \right) + v^*(x) e^x u'(x) \bigg|_0^1 =
\int_0^1 dx \left( (e^x v^*(x))'' u(x) + (x^2 v^*(x))' u(x) + v^*(x) u(x) \right)
+ v^*(x) e^x u'(x) \bigg|_0^1 - (e^x v^*(x))' u(x) \bigg|_0^1 =
\int_0^1 dx u(x) \left( e^x \frac{d^2}{dx^2} + (2e^x + x^2) \frac{d}{dx} + 2x + 1 \right) v^*(x) +
+ v^*(x) e^x u'(x) \bigg|_0^1 - (e^x v^*(x))' u(x) \bigg|_0^1
\]

The boundary terms with \(u(0) = u(1) = 0\) will vanish if
\[
v^*(0) e^x u'(0) = v^*(1) e^x u'(1) = 0
\]

which holds for \(v(0) = v(1) = 0\). These are the adjoint boundary conditions. The adjoint operator can be read off of the above
\[
L^\dagger_x = e^x \frac{d^2}{dx^2} + (2e^x + x^2) \frac{d}{dx} + 2x + 1
\]

3.) Consider a second order linear differential equation of the form
\[
\frac{d^2}{dx^2} f(x) + p(x) \frac{d}{dx} f(x) + q(x) f(x) = 0
\]

with two independent solutions \(f_1(x)\) and \(f_2(x)\).

a. Show that the Wronskian of this differential operator is never zero.

\[
W(x) = f_1(x) f_2'(x) - f_1'(x) f_2(x)
\]

Differentiating this assuming that both functions satisfy the differential equation gives
\[
\frac{dW(x)}{dx} = -p(x) W(x)
\]

which can be solved
\[
W(x) = ce^{-\int_0^x p(x') dx'} \neq 0
\]
b. Show there can be at most two linearly independent solutions to this equation.

The simplest way to see this is to assume that there are three. Then the functions and their first and second derivatives should be independent. This requires

$$\det \begin{pmatrix} u_1 & u_2 & u_3 \\ u_1' & u_2' & u_3' \\ u_1'' & u_2'' & u_3'' \end{pmatrix} \neq 0$$

but using the differential equation it follows that the third row is a linear combination of the first two rows, so the determinant must be 0.

4.) Use the series method to solve the second order differential equation with constant coefficients,

$$L_x|f\rangle = 0$$

where

$$L_x = \frac{d^2}{dx^2} + a \frac{d}{dx} + b$$

with boundary conditions

$$\langle 0|f \rangle = 1$$

$$\frac{d}{dx}\langle x|f \rangle |_{x=0} = 1$$

Express $f(z)$ as

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

Using this in the differential equation gives

$$\sum_{n=0}^{\infty} \left( n(n-1)z^{n-2} + anz^{n-1} + bz^n \right) f_n$$

Rewriting this as an expression in powers of $z$ gives

$$\sum_{m=0}^{\infty} \left( (m+2)(m+1)f_{m+2} + a(m+1)f_{m+1} + bf_m \right) z^m = 0$$

where $f_0 = 1$ and $f_1 = 1$ are fixed by boundary conditions. This gives

$$f_{n+2} = -\frac{a(m+1)}{(m+1)(m+2)}f_{m+1} - \frac{b}{(m+1)(m+2)}f_m =$$

$$= -\frac{a}{(m+2)}f_{m+1} - \frac{b}{(m+1)(m+2)}f_m$$

which expresses the higher order coefficients in term of the lower order ones.
If you convert this to a first order system
\[ f'(x) = g(x) \quad g'(x) = -ag(x) - bf \]
if the let \( f(z) = (f(z), g(z)) \) the equation can be expressed in matrix form as
\[ f'(z) = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} f(z) = Mf \]
In this case we assume
\[ f'(z) = \sum_n f_n z^n \]
Using this in the differential equation gives
\[ \sum_{n=0}^{\infty} n f_n z^{n-1} = \sum_n Mf_n z^n \]
or
\[ f_n = \frac{1}{n} Mf_{n-1} \quad n > 0 \]
which has the solution
\[ f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} M^n f(0). \]
that can be solved in closed for by diagonalizing the \( 2 \times 2 \) matrix \( M \).

b. What is the domain of analyticity of your solution? The origin is an ordinary point and the constant coefficient are entire so the solution is an entire function

5.) Given a Strum-Liouville operator of the form
\[ L = \frac{d}{dx} g(x) \frac{d}{dx} - h(x) \]
with weight \( w(x) = 1 \), \( x \in [a, b] \), and \( g(x) \) and \( h(x) \) real. Show explicitly that the eigenvalues of \( L|f_n\rangle = \lambda_n |f_n\rangle \) are real and that eigenvectors corresponding different eigenvalues are orthogonal on \([a, b]\)

Strum Liouville means
\[ (v, L_x u) = (L_x v, u). \]
First if \( L_x u = \lambda_u u \) and \( L_x v = \lambda_v v \)
\[ \lambda_u (v, u) = (v, L_x u) = (L_x v, u) = \lambda_v^* (v, u) \]
or
\[ (\lambda_u - \lambda_v^*) (v, u) = 0. \]
If \( u = v \) then this means that the eigenvalues are real, while if \( u \neq v \) they must be orthogonal if they have different eigenvalues.
6.) Consider the differential operator

\[ L = x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} + x^2 \]

on the interval \([0, \infty]\). Find a weight function that makes this a self-adjoint operator. Note

\[ L = x \frac{d}{dx} \left( x \frac{d}{dx} \right) + x^2 \]

It follows that

\[
\int \frac{dx}{x} v^*(x) \left( x \frac{d}{dx} \left( x \frac{du(x)}{dx} \right) + x^2 u(x) \right) = \\
\int \left( -\frac{dv^*(x)}{dx} x \frac{du(x)}{dx} + v^*(x) x^2 u(x) \right) + v^*(x) x \frac{du(x)}{dx} \bigg|_{0}^{\infty} = \\
\int \left( u(x) \frac{d}{dx} \frac{dv^*(x)}{dx} + v^*(x) x^2 u(x) \right) + dv^*(x) x \frac{du(x)}{dx} \bigg|_{0}^{\infty} \left( \frac{dv^*(x)}{dx} x u(x) \right)_{0}^{\infty} = \\
\int \left( \frac{1}{x} u(x) \frac{xd}{dx} \frac{dv^*(x)}{dx} + v^*(x) x^2 u(x) \right) + dv^*(x) x \frac{du(x)}{dx} \bigg|_{0}^{\infty} \left( \frac{dv^*(x)}{dx} x u(x) \right)_{0}^{\infty} = \\
\int \frac{dx}{x} u(x) (L_x v(x))^* + dv^*(x) x \frac{du(x)}{dx} \bigg|_{0}^{\infty} - \frac{dv^*(x)}{dx} x u(x) \bigg|_{0}^{\infty} = \\
\int (L_x v(x))^* + dv^*(x) x \frac{du(x)}{dx} \bigg|_{0}^{\infty} - \frac{dv^*(x)}{dx} x u(x) \bigg|_{0}^{\infty} = \\
\int \frac{dx}{x} u(x) (L_x v(x))^* + dv^*(x) x \frac{du(x)}{dx} \bigg|_{0}^{\infty} - \frac{dv^*(x)}{dx} x u(x) \bigg|_{0}^{\infty} = \\
\int \frac{dx}{x} u(x) (L_x v(x))^* + dv^*(x) x \frac{du(x)}{dx} \bigg|_{0}^{\infty} - \frac{dv^*(x)}{dx} x u(x) \bigg|_{0}^{\infty} =
\]

The boundary terms are symmetric. In this case the weight is \(1/x\)