Topic 1: Constructing a second solution to the homogeneous equation given a first solution

Let \( u_1(x) \) be a solution of the homogeneous differential equation:

\[
a(x)u''_1(x) + b(x)u'_1(x) + c(x)u_1(x) = 0.
\]

Look for another solution of the form \( u_2(x) = h(x)u_1(x) \). Requiring \( u_2(x) \) to satisfy the differential equation gives

\[
h(x)(a(x)u''_1(x) + b(x)u'_1(x) + c(x)u_1(x)) +
\]

\[
h'(x)(2a(x)u'_1(x) + b(x)u_1(x)) + h''(x)a(x)u_1(x) = 0
\]

The coefficient of \( h(x) \) vanishes because \( u_1(x) \) is a solution of the differential equation. This gives the following equation for \( h'(x) \)

\[
\frac{h''(x)}{h'(x)} = -\frac{1}{u_1(x)a(x)} (2a(x)u'_1(x) + b(x)u_1(x)) = -\frac{2u'_1(x)}{u_1(x)} - \frac{b(x)}{a(x)}.
\]

In order to integrate this equation define \( p(x) \) by

\[
\frac{dp}{dx} = \frac{b(x)}{a(x)}
\]

This can be integrated to find \( p(x) \):

\[
p(x) = e^{\int_{x_0}^{x} \frac{b(x')}{a(x')} dx'}.
\]

The equation for \( h'(x) \) in terms of \( p(x) \) becomes

\[
\frac{h''(x)}{h'(x)} = -\frac{2u'_1(x)}{u_1(x)} - \frac{p'(x)}{p(x)}.
\]

Integrating this equation gives

\[
h'(x) = \frac{1}{u_1(x)^2p(x)} = e^{-\int_{x_0}^{x} \frac{b(x')}{a(x')} dx'} \frac{1}{u_1(x)^2}.
\]

where we have used the expression for \( p(x) \). Integrating once again gives

\[
h(x) = \int_{x_0}^{x} dx' e^{-\int_{x_0}^{x'} \frac{b(x'')}{a(x''')} dx''} \frac{1}{u_1(x')^2}
\]

and an expression for the other solution:

\[
u_2(x) = u_1(x)h(x) = u_1(x) \int_{x_0}^{x} dx' e^{-\int_{x_0}^{x'} \frac{b(x'')}{a(x''')} dx''} \frac{1}{u_1(x')^2}
\]
The Wronskian for this pair of solutions is

\[ W = \begin{vmatrix} u_1 & u_1^h \\ u'_1 & u'_1 h + h'u_1 \end{vmatrix} = u_1^2 h' = \frac{1}{p(x)} = e^{-\int_{x_0}^{x} \frac{b(x')}{a(x')} dx'} > 0 \]  

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Since this is never 0 these solutions are linearly independent.

Remark: all of the classical orthogonal polynomials are solutions of second order differential equations. The second solution normally is not a polynomial. This method can be used to construct the non-polynomial solutions from the polynomial solutions.