### Topic 4: Calculating Green’s functions

Green functions are only useful if they can be computed. Since any second order differential operator with real coefficients is a self-adjoint operator on a space Hilbert space with weight \( w(x) \), in that case the Green’s function is a solution to

\[
L_x G(x, y) = \frac{\delta(x - y)}{w(x)} \tag{1}
\]

When \( x \neq y \) the right-side of this equation is zero and \( G(x, y) \), as a function of \( x \), is a solution of the homogeneous differential equation.

\[
(a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} + c(x)) G(x, y) = 0 \quad x \neq y
\]

Using

\[
\frac{1}{p(x)} \frac{dp(x)}{dx} = \frac{b(x)}{a(x)} \quad p(x) = e^{\int_{x_0}^{x} \frac{b(x')}{a(x')} dx'}
\]

this equation becomes, including the delta-function term:

\[
\frac{d}{dx} \left( p(x) \frac{dG(x, y)}{dx} \right) + \frac{p(x)c(x)}{a(x)} G(x, y) = \frac{\delta(x - y)p(x)}{a(x)w(x)}.
\]

Integrating both sides of this equation over a small volume \((y - \epsilon, y + \epsilon)\) gives

\[
(p(y^+) \frac{dG(y^+, y)}{dx}) - (p(y^-) \frac{dG(y^-, y)}{dx}) + \int_{y-\epsilon}^{y+\epsilon} c(x)p(x) \frac{dx}{a(x)} G(x, y) dx = \frac{p(y)}{a(y)w(y)}
\]

Taking the limit \( \epsilon \to 0 \) gives

\[
\frac{dG(y^+, y)}{dx} - \frac{dG(y^-, y)}{dx} = \frac{1}{a(y)w(y)} \tag{2}
\]

If \( G(x, y) \) is bounded near \( x = y \) then the derivative of \( G(x, y) \) with respect to \( x \) has a discontinuity at \( x = y \). Since the discontinuity is finite, integration over the discontinuity is continuous which implies that \( G(x, y) \) is continuous at \( x = y \).

The solution (1) involves using linear combinations of of the independent solutions of the homogeneous equations that satisfy the boundary conditions at \( a \) and \( b \) and have the discontinuity (2) at \( x = y \). In general there will be two solutions for \( a \leq x \leq y \) and two more for \( y \leq x \leq b \).

\[
G(x, y) = \begin{cases} 
\alpha(y)u_1(x) + \beta(y)u_2(x) & x < y \\
\gamma(y)u_1(x) + \delta(y)u_2(x) & x > y 
\end{cases}
\]

where

\[
\alpha(y)u_1(y) + \beta(y)u_2(y) = \gamma(y)u_1(y) + \delta(y)u_2(y)
\]
\[ \gamma(y)u_1'(y) + \delta(y)u'_2(y) - \alpha(y)u'_1(y) - \beta(y)u'_2(y) = \frac{1}{a(y)w(y)} \]

and \( G(x, y) \) and \( \frac{\partial G(x, y)}{\partial x} \) satisfy the same boundary conditions as \( u(x), u'(x) \) at \( x = a \) and \( b \).

This assumes that \( L_x \) has an inverse. This will happen if the boundary conditions uniquely fix all of coefficients. This can fail if there are non-zero solutions to \( L_x u(x) = 0 \) that satisfy both boundary conditions. Then it is necessary to use a more general construction.