Topic 2: example 1

It is useful to illustrate this construction with some examples. The first one is trivial:

\[ L_x = \frac{d^2}{dx^2} \quad u(0) = u(a) = 0 \]

In this case we take \( w(x) = 1 \). The Green’s function satisfies

\[ \frac{d^2 G(x, y)}{dx^2} = \delta(x - y) \]

In this case the independent solutions are

\[ u_1(x) = 1 \quad u_2(x) = x \]

\[ G(x, y) = \begin{cases} 
\alpha(y) + \beta(y)x & 0 < x < y \\
\gamma(y) + \delta(y)x & y < x < a 
\end{cases} \]

Boundary conditions at 0 give \( \alpha(y) = 0 \). The boundary condition at \( x = a \) gives \( \gamma(y) = -a\delta(y) \). Continuity at \( y \) gives

\[ y\beta(y) = \delta(y)(y - a) \]

while the discontinuity of \( G(x, y) \) at \( y \) gives

\[ \delta(y) - \beta(y) = 1 \]

Solving gives

\[ \beta(y) = (y - a)/a \quad \delta(y) = \frac{y}{a} \]

so the Green’s function for this operator (including boundary conditions) is

\[ G(x, y) = \begin{cases} 
(y - a)x/a & 0 < x < y \\
(x - a)y/a & y < x < a 
\end{cases} \]

If we take \( f(x) = 1 \)

\[ u(x) = \int_0^a G(x, y)dy = \]

\[ \int_0^x G(x, y)dy + \int_x^a G(x, y)dy = \]

\[ \int_0^x (x - a)yady + \int_x^a (y - a)xady = \]

\[ x^2 - (x^2/2 - ax)x/a + (a^2/2 - a^2)x/a = \]

\[ x^2/2 - ax/2 \]

This satisfies the inhomogeneous equation and the boundary conditions. Note that the only linear combination of the form \( u(x) = ax + b \) satisfying \( u(0) = u(a) = 0 \) is the trivial zero solution.