In this example we convert a differential operator $L_x$ with a weight $w(x)$ into an equivalent problem with $w(x) = 1$. To do this replace the differential operator $L_x$ with $w(x) L_x$. This product is self adjoint with respect to a norm with weight 1 provided $L_x$ is self adjoint with respect to weight $w(x)$.

We start with the original differential operator

$$L_x = \frac{1}{w(x)} \frac{d}{dx} \left( p(x) \frac{d}{dx} \right) + c(x).$$

In this case the inhomogeneous differential equation

$$w(x) L_x = f(x)$$

has the form

$$\frac{d}{dx} \left( p(x) \frac{du(x)}{dx} \right) + w(x) c(x) u(x) = f(x) \quad a \leq x \leq b$$

with homogeneous boundary conditions

$$u(a) = \alpha \frac{du(a)}{dx}, \quad u(b) = \beta \frac{du(b)}{dx}.$$  

We let $u_1(x)$ and $u_2(x)$ be independent solutions of the homogeneous version of the differential equation. The Green’s function has the form

$$G(x, y) = \begin{cases} a_{11}(y) u_1(x) + a_{12}(y) u_2(x) & a \leq x < y \\ a_{21}(y) u_1(x) + a_{22}(y) u_2(x) & y < x \leq b \end{cases}$$

First we apply the boundary conditions at $a$ and $b$:

$$a_{11}(y) u_1(a) + a_{12}(y) u_2(a) = \alpha (a_{11}(y) u_1'(a) + a_{12}(y) u_2'(a))$$

$$a_{21}(y) u_1(b) + a_{22}(y) u_2(b) = \beta (a_{21}(y) u_1'(b) + a_{22}(y) u_2'(b))$$

The solution in each region has the form

$$U_<(x) := u_1(x) + \frac{a_{12}}{a_{11}} u_2(x) =$$

$$u_1(x) - \frac{u_1(a) - \alpha u_1'(a)}{u_2(a) - \alpha u_2'(a)} u_2(x)$$

$$U_>(x) := u_1(x) + \frac{a_{22}}{a_{21}} u_2(x) =$$

$$u_1(x) - \frac{u_1(b) - \beta u_1'(b)}{u_2(b) - \beta u_2'(b)} u_2(x)$$

These also satisfy the boundary condition in each region.
Continuity at $x = y$ gives
\[ a_{11}(y)U_<(y) = a_{21}(y)U_>(y) \]
while the discontinuity at $y$ gives
\[ a_{21}(y)U'_>(y) - a_{11}(y)U'_<(y) = \frac{1}{p(y)} \]
(recall that in this case $a(x)w(x) = p(x)$)
These equations can be solved to find the coefficients
\[ a_{11}(y) = \frac{U_>(y)}{p(y)(U'_>(y)U_<(y) - U'_<(y)U_>(y))} \]
\[ a_{21}(y) = \frac{U_<(y)}{p(y)(U'_>(y)U_<(y) - U'_<(y)U_>(y))} \]
This gives
\[ G(x, y) = \begin{cases} \frac{U_>(y)U_<(x)}{p(y)(U'_>(y)U_<(y) - U'_<(y)U_>(y))} & a \leq x < y \\ \frac{U_<(y)U_>(x)}{p(y)(U'_>(y)U_<(y) - U'_<(y)U_>(y))} & y < x \leq b \end{cases} \]
The denominator is the Wronskian of the solutions $U_>(x)$ and $U_<(x)$ of the homogeneous equation.
From the differential equation
\[ \frac{d}{dx}(p(x)\frac{d}{dx}(U_<(x)U_>(x))) = \]
\[ \frac{d}{dx}(p(x)(U'_<(x)U_>(x) - U_<(x)U'_>(x))) = \]
\[ U_>(x)\frac{d}{dx}(p(x)U'_<(x)) - U_<(x)\frac{d}{dx}(p(x)U'_>(x)) = \]
\[ -(U_<(x)U_>(x))(w(x)c(x)p(x) - w(x)c(x)p(x)) + f(x) - f(x) = 0 \]
This means that
\[ p(x)(U'_<(x)U_>(x) - U_<(x)U'_>(x)) \]
is a constant. If $U_<(x)$ and $U_>(x)$ are independent solutions then their Wronskian is non-zero and the constant must be non-zero.
Conversely if the constant is zero then
\[ U_<(x) = \eta U_>(x) \]
Because these functions are proportional they satisfy the same homogeneous boundary conditions.