Lecture 26 - Topic 3: Inhomogeneous boundary conditions

The class of boundary conditions that we have studied so far are homogeneous boundary conditions. They have the form

\[ B(u) := c_1 u(a) + c_2 u(b) + c_3 u'(a) + c_4 u'(b) = 0 \]

where the \( c_i \) are constants. It is also possible to impose boundary conditions of the form

\[ B(u) := c_1 u(a) + c_2 u(b) + c_3 u'(a) + c_4 u'(b) = d \neq 0. \]

These are called inhomogeneous boundary conditions. The difference is that linear combinations of functions satisfying inhomogeneous boundary conditions do not satisfy the same boundary conditions. However, adding solutions that satisfy the homogeneous boundary conditions to a solution satisfying the inhomogeneous boundary conditions results in a class of functions satisfying the inhomogeneous boundary conditions.

To treat problems with inhomogeneous boundary conditions we let

\[ v(x) = g(x, y) = G(x, y) \]

where \( G(x, y) \) the Green’s function associated with the homogeneous boundary conditions. We recall that if the coefficients of the differential equation are real the weight function can be chosen so \( L_x \) is Hermitian with respect to the weight. This means the Green’s function and its adjoint are the same.

We use this \( v(x) \) in the generalized Green’s identity where we do not assume that \( u(x) \) satisfies the homogeneous boundary conditions

\[ \int_a^b w(x)(L_x G(x, y) u(x) - G(x, y) L_x u(x))dx = \]

\[ \frac{1}{p(x)}(\frac{dG(x, y)}{dx}u(x) - G(x, y)u'(x))|_{x=a}^{x=b}. \]

We assume that \( L_x u(x) = f(x) \). Using

\[ L_x G(x, y) = \frac{1}{w(x)}\delta(x - y) \]

and \( L_x u(x) = f(x) \) in the generalized Green’s identity gives

\[ u(y) = \int G(x, y)w(x)f(x)dx + \frac{1}{p(x)}(\frac{dG(x, y)}{dx}u(x) - G(x, y)u'(x))|_{x=a}^{x=b}. \]

So far we have not applied boundary conditions to \( u(x) \). There are four unknowns - the homogeneous boundary conditions satisfied by \( G(x, y) \) eliminate two of the unknowns. The inhomogeneous boundary conditions can be used in this expression to fix the remaining values of \( u(x) \) and \( u'(x) \) on the boundary.
An example is the best way to understand how this works. We look for a solution of
\[ u''(x) = f(x) \]
on \( [0, a] \) with inhomogeneous boundary conditions \( u(0) = \sigma_1 \) and \( u(a) = \sigma_2 \). The corresponding homogeneous boundary conditions are \( u(0) = u(a) = 0 \).

The independent solutions of the homogeneous equation are 1 and \( x \). The Green function is a linear combination of 1 and \( x \) satisfying the homogeneous boundary conditions, continuous at \( x = y \) and has a derivative with a discontinuity of 1 at \( x = y \). This was computed previously:
\[ G(x,y) = \begin{cases} (y-a)x/a & x < y \\ (x-a)y/a & y < x \end{cases} \]

Note that \( g(x,y) = G(x,y) \) (This follows because \( x \leftrightarrow y \), but the inequalities \( x < y \) and \( x > y \) also reverse). Inspection shows that \( G(x,y) \) vanishes at \( x = y \) and the derivative has a discontinuity of 1 at \( x = y \). Note that
\[ \frac{dG(x,y)}{dx} \begin{cases} (y-a)/a & x < y \\ y/a & x > y \end{cases} \]

Solving the inhomogeneous equation for \( f(x) = 1 \) gives
\[ u(y) = \int_{0}^{y} (y-a)x/adx + \int_{y}^{a} (x-a)y/adx + u(a)(y/a) - u(0)(y/a-1) - (G(x,y)u'(x))|_{x=a=0}. \]
The Green’s function \( G(x,y) = 0 \) at \( x = 0 \) and \( x = a \) which eliminates the coefficients of \( u'(0) \) and \( u'(a) \). This gives
\[ u(y) = \int_{0}^{y} (y-a)x/adx + \int_{y}^{a} (x-a)y/adx(\sigma_2 - \sigma_1)(x/a) + \sigma_1 = \]
\[ y^2/2 - ay/2 + (\sigma_2 - \sigma_1)(y/a) + \sigma_1. \]
It is easy to check that this function has all of the required properties.