Lecture 28 - Topic 1: Integral representations

An integral representation of a function $u(z)$ has the form

$$u(z) = \int_a^b K(z, t) v(t) dt$$

along some path. If $u(z)$ is a solution of the differential equation satisfying boundary conditions in some region $D$, and this solution agrees with a series solution defined in a region $R$ where $R \cap D$ includes the neighborhood of a point, then the solutions, both being analytic functions must agree. It often happens that using the integral representation extends the domain of analyticity. We found this with the Gamma function, where we had two integral representations, and one had a larger domain of analyticity.

Example 1: Consider the series

$$\sum_{n=0}^{\infty} z^n.$$ 

This converges to $1/(1 - z)$ for $|z| < 1$, where it is analytic in this region. Consider the integral

$$f(z) := \int_0^{\infty} e^{zt} e^{-t} dt.$$ 

This is analytic for $Re(z) < 1$ which includes large negative values of the real part of $z$. Expanding the exponent

$$f(z) = \int_0^{\infty} e^{zt} e^{-t} dt = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_0^{\infty} t^n e^{-t} dt = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Gamma(n + 1) = \sum_{n=0}^{\infty} z^n.$$ 

These two analytic functions agree for $|z| < 1$, so this integral representation extends the domain of analyticity from $|z| < 1$ to $Re(z) < 1$.

Example 2: Consider the differential equation

$$L_z u(z) = 0.$$ 

Assume that we have found $K(z, t)$ and a differential operator $M_t$ in $t$ satisfying

$$L_z K(z, t) = M_t K(z, t).$$

Then

$$L_z u(z) = \int_a^b L_z K(z, t) v(t) dt = \int_a^b M_t K(z, t) v(t) dt.$$ 

If $M_t^\dagger$ is adjoint of $M_t$ then

$$v(t) M_t K(z, t) - K(z, t) M_t^\dagger v(t) = \frac{\partial}{\partial t} Q(K(z, t) v(t)).$$
Integrating $t$ from $a$ to $b$ gives

$$L_z u(z) = \int_a^b L_z K(z, t)v(t)dt = \int_a^b K(z, t)M_t^1 v(t) + Q(K(z, t)v(t))|_{t=a}^{t=b}. $$

This will vanish provided $Q(K(z, t)v(t)$ vanishes at the endpoints and $v(t)$ is a solution of the adjoint equation

$$M_t^1 v(t) = 0. $$

It follows that

$$u(z) = \int_a^b K(z, t)v(t)dt$$

is a solution to the equation since

$$L_z u(z) = \int_a^b L_z K(z, t)v(t)dt = \int_a^b v(t)M_t K(z, t)dt = \int_a^b K(z, t)M_t^1 v(t)dt = 0. $$

The general method is given $L_z$, find $M_t$, solve for homogeneous solutions of the adjoint equation with boundary conditions that make the boundary term $Q(K(z, t)v(t))|_{t=a}^{t=b}$ vanish. Several examples will be given in the material that follows.