Lecture 28 - Topic 3: Fuchsian equations with three regular points

In this section we show that class of equations with three singularities can all be related to solutions of an equation called the hypergeometric differential equation. It turns out that many equations that appear in physics problems are related to the hypergeometric function, which is analytic solution of this equation.

Consider differential equations of the form

\[ \frac{d^2 u(z)}{dz^2} + \left( \frac{1 - \alpha - \alpha'}{z - z_1} + \frac{1 - \beta - \beta'}{z - z_2} + \frac{1 - \gamma - \gamma'}{z - z_3} \right) \frac{du(z)}{dz} + \]

\[ \left( \frac{(z_1 - z_2)(z_1 - z_3)\alpha\alpha'}{z - z_1} + \frac{(z_2 - z_1)(z_2 - z_3)\beta\beta'}{z - z_2} + \frac{(z_3 - z_1)(z_3 - z_2)\gamma\gamma'}{z - z_3} \right) \frac{u(z)}{(z - z_1)(z - z_2)(z - z_3)} \]

where

\[ \alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1 \]

this equation has regular singularities at \( z = z_1, z = z_2 \) and \( z = z_3 \). The form of the equation is chosen so the indicial equation about each singular point has the form

\[ \begin{align*}
    r(r - 1) + (1 - \alpha - \alpha')r + \alpha\alpha' &= (r - \alpha)(r - \alpha') - 0 \\
    r(r - 1) + (1 - \beta - \beta')r + \beta\beta' &= (r - \beta)(r - \beta') - 0 \\
    r(r - 1) + (1 - \gamma - \gamma')r + \gamma\gamma' &= (r - \gamma)(r - \gamma') - 0 
\end{align*} \]

which shows that the roots are \( \alpha \) and \( \alpha' \), \( \beta \) and \( \beta' \), and \( \gamma \) and \( \gamma' \). (you should check this).

This equation is called the equation of Riemann. The solutions are represented by

\[ u(z) = P \begin{bmatrix} z_1 & z_2 & z_2 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{bmatrix} \]

where \( z_i \) represent the singular points, followed by the roots of the indicial equation in each column. This is called a P-symbol. As it is defined it represents a general solution to the differential equation.

This solution has 9 parameters, \( z_1, z_2, z_3, \alpha, \beta, \gamma \) and \( \alpha', \beta', \gamma' \). This can be reduced to three parameters by defining

\[ v(z) = (z - z_1)^r(z - z_2)^s(z - z_3)^t u(z) \quad r + s + t = 0 \]

\[ v(z) = (z - z_1)^r(z - z_2)^s(z - z_3)^t P \begin{bmatrix} z_1 & z_2 & z_2 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{bmatrix} \]

After some algebra this becomes the solution of another equation of the same form:

\[ v(z) = P \begin{bmatrix} z_1 & z_2 & z_2 \\ \alpha + r & \beta + s & \gamma + t \\ \alpha' + r & \beta' + s & \gamma' + t \end{bmatrix} \]
We can also make a variable change using the homographic transformation
\[ z' = \frac{Az + B}{Cz + D}, \quad z_i' = \frac{Az_i + B}{Cz_i + D}. \]

This gives
\[ P \begin{cases} z_1 & z_2 & z_2 \\ \alpha & \beta & \gamma & z \end{cases} = P \begin{cases} z_1' & z_2' & z_2' \\ \alpha' & \beta' & \gamma' \end{cases} \]
which gives another equation of the same form with different regular singular points. The verification of these formulas is tedious but uses only algebra and differentiation.

To express all of these solutions in a standard form we first use the homographic transformation to transform the singular points to \( z_1' = 0, z_2' = \infty, z_3' = 1 \). This requires
\[
\begin{align*}
A &= \frac{z_3 - z_2}{z_2(z_1 - z_3)} & B &= \frac{z_1(z_2 - z_3)}{z_2(z_1 - z_3)} & C &= -\frac{1}{z_2}
\end{align*}
\]
which gives the variable transformation:
\[ z' = \frac{(z_3 - z_2)(z - z_1)}{(z_3 - z_1)(z - z_2)} \]
and
\[ P \begin{cases} z_1 & z_2 & z_2 \\ \alpha & \beta & \gamma & z \end{cases} = P \begin{cases} 0 & \infty & 1 \\ \alpha' & \beta' & \gamma' \end{cases} \]
which replaces the 9 parameter expression by a 6 parameter expression. Next we use the transformation (1) with
\[ r = -\alpha, \quad s = \alpha + \gamma, \quad t = -\gamma \]
which gives
\[ P \begin{cases} z_1 & z_2 & z_2 \\ \alpha & \beta & \gamma & z \end{cases} = \begin{pmatrix} z - z_1 \\ z - z_2 \end{pmatrix} \begin{pmatrix} z_1 & z_2 & z_2 \\ \alpha & \beta & \gamma \end{pmatrix} = P \begin{cases} 0 & \infty & 1 \\ a & b & c - a - b \end{cases} \]
where
\[ a = \alpha + \beta + \gamma, \quad b = \alpha + \beta' + \gamma, \quad c = 1 + \alpha - \alpha' \]
The key result is that any \( P \) symbols can be expressed in terms of the following 3-parameter \( P \) symbol:
\[ P \begin{cases} 0 & \infty & 1 \\ 0 & a & 0 \end{cases} = \begin{pmatrix} 1 - c \\ b \end{pmatrix} \begin{cases} 0 & \infty & 1 \\ a & 0 \end{cases} \begin{pmatrix} 1 - c \\ b \end{pmatrix} \]
(2)
Making these substitutions in the original differential equations, the equation satisfied by (2) becomes

\[
z(z - 1) \frac{d^2u}{dz^2} + (c - (a + b + 1)z) \frac{du}{dz} - abu(z) = 0
\]

This equation is called hypergeometric equation. It still has 2 independent solutions. The solution that is analytic at the origin is denoted by \( F(a, b, a; z) \). It is called the hypergeometric function.

Putting everything together shows that one solution of a general symbol is relate to the hypergeometric function as follows

\[
P \left\{ \begin{array}{ccc} z_1 & z_2 & z_2 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{array} \right\} = \left( \frac{z - z_1}{z - z_2} \right)^\alpha \left( \frac{z - z_3}{z - z_2} \right)^\gamma F \left( \alpha + \beta + \gamma, \alpha + \beta' + \gamma, 1 - \alpha' ; \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_2)} \right)
\]

This shows that it is possible to express all of these solutions in terms of solutions of the hypergeometric equation.