Lecture 28 - Topic 4: The Hypergeometric equation

This topic is concerned with solutions of the hypergeometric equation:

\[ z(z-1)\frac{d^2u(z)}{dz^2} + (c - (a + b + 1)z)\frac{du(z)}{dz} - abu(z) = 0 \]  \hspace{1cm} (1)

This equation has singularities at 0, 1 and \( \infty \). In general the solutions will be multi-valued. We choose a branch cut running from 1 \( \rightarrow \) \( \infty \) along the positive real axis. The analytic solution in the neighborhood of \( z_0 = 0 \) has the form

\[ F(a, b, c; z) = \sum_{n=0}^{\infty} f_n z^n \]

where substituting the series in (1) gives the following equations for the expansion coefficients \( f_n \)

\[ f_n = \frac{(a + n - 1)(b + n - 1)}{n(c + n - 2)} f_{n-1} \quad f_0 := 1 \]

This gives a convergent series provided \( c \) is not a non-negative integer. In that case the solution is represented by the series

\[ F(a, b, c; z) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)n!} z^n. \]

Since there is pole at \( z = 1 \) this series converges uniformly for \( |z| < 1 \).

When \( c \) is not a non-negative integer, then the second solution has the form

\[ u_2(z) = z^r g(z) = z^{1-c}g(z). \]

Substituting this in (1)

\[ z(z-1)\frac{d^2g(z)}{dz^2} + (c - (a + b + 1)z)\frac{dg(z)}{dz} - abg(z) = 0 \]

after performing algebra, gives the following differential equation for \( g(z) \):

\[ z(z-1)\frac{d^2g(z)}{dz^2} + (c - 2 + (a + b - 2c + 3)z)\frac{dg(z)}{dz} + (a - c + 1)(b - c + 1)g(z) = 0 \]

This is the hypergeometric equation for

\[ F(b - c + 1, a - c + 1, 2 - c, z). \]

It follows that an independent second solution of the hypergeometric different equation in a neighborhood of the origin is

\[ u_2(z) = z^{1-c}F(b - c + 1, a - c + 1, 2 - c, z) \]

This one generally has a branch cut due to the factor \( z^{1-c} \).

In general there are two solutions associated with each of the three regular singular points. For the next topic we show how they are also related to \( F(a, b, a; z) \)