Lecture 29 - Topic 1: The Hypergeometric equation

In the previous lecture we constructed two independent solutions of the hypergeometric differential equation. One was analytic in a neighborhood of the origin and the other had an isolated singularity at \( z = 0 \), but both could be expressed in terms of the analytic solution (the hypergeometric function).

There are also solutions of the equation in the neighborhood of the other two singular points. The differential equation associated with a given \( P \) symbol remains the same under exchangint the columns of the \( P \) symbol and invariant under the interchange of the roots of the indicial equation \( \alpha \leftrightarrow \alpha', \beta \leftrightarrow \beta', \gamma \leftrightarrow \gamma' \). There are \( 48 = 3 \times 2^3 \) such permutations that leave the equation unchanged.

On the other hand the expression on the right hand side of

\[
P\left\{ \begin{array}{ccc}
    z_1 & z_2 & z_2 \\
    \alpha & \beta & \gamma \\
    \alpha' & \beta' & \gamma'
\end{array} \right\} = \left( \frac{z-z_1}{z-z_2} \right)^\alpha \left( \frac{z-z_3}{z-z_2} \right)^\gamma F\left( \alpha + \beta + \gamma, \alpha + \beta' + \gamma, 1 - \alpha - \alpha'; \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_2)} \right)
\]

where the \( z_i \) are \( \{0, 1, \infty\} \) and the roots of the indicial equations are \( \alpha = 0, \alpha' = 1 - c, \beta = a, \beta' = b, \gamma = 0 \) and \( \gamma' = c-a-b \) do not have these symmetries, although it is invariant under \( a \leftrightarrow b \). These transformations generate \( 48/2 = 24 \) solutions of the hypergeometric differential equation that can all be expressed directly in terms of the hypergeometric function.

Without going into the details, these relations can be used to express the analytic and singular solution about the other singular points, \( z_0 = 1 \) and \( z_0 = \infty \) in terms of the hypergeometric function. The expressions are

\[
G_{11}(z) = F(a, b, a + b + 1 - c; 1 - z)
\]
\[
G_{12}(1-z)^{c-a-b}F(c - b, c - a, 1 + c - a - b; 1 - z)
\]
\[
G_{\infty 1}(z) = z^{-a}F(a, a - c + 1, a - b + 1; \frac{1}{z})
\]
\[
G_{\infty 2}(z) = z^{-b}F(b, b - c + 1, b - a + 1; \frac{1}{z})
\]

The important observation is that all of the solutions of interest can be expressed directly in terms of the hypergeometric function.