Lecture 29 - Topic 2: Integral representations of the hypergeometric function

The radius of convergence of the series expansion of the hypergeometric function about \( z = 0 \) is 1, due to the singular point at 1. This can be extended using integral representations.

The hypergeometric equation has the form of equations where an integral representation can be constructed using a kernel of the Euler form:

\[
\int_C (z-t)^\lambda v(t)dt
\]

where we use the hypergeometric equation to determine \( \lambda \), \( v(t) \) and the contour \( C \). The hypergeometric equation is

\[
z(1-z)u''(z) + [c-z(a+b+1)]u'(z) - abu(z) = 0.
\]

The first step in the construction is to insert the integral representation into the differential equation. The derivatives are

\[
u'(z) = \int_C (z-t)^{\lambda-1}v(t)dt
\]

\[
u''(z) = \int_C \lambda(z-t)^{\lambda-2}v(t)dt
\]

Using these expressions for the derivatives in the differential equation gives

\[
\int_C \left( z(1-z)\lambda(z-1)(z-t)^{\lambda-2} + [c-z(a+b+1)]\lambda(z-t)^{\lambda-1} - ab(z-t)^\lambda \right)v(t)dt = 0.
\]

Next we factor out \((z-t)^{\lambda-2}\) to get

\[
\int_C (z(1-z)\lambda(z-1)+ [c-z(a+b+1)]\lambda(z-t) - ab(z-t)^2)(z-t)^{\lambda-2}v(t)dt = 0.
\]

Next we express the coefficient as a polynomial in powers of \( z \)

\[
0 = \int_C \left( z^2(-\lambda^2 + \lambda - \lambda(a+b+1) - ab) + z(\lambda^2 - \lambda + c\lambda + t\lambda(a+b+1) + 2abt) + (-c\lambda t - abt^2) \right) (z-t)^{\lambda-2}v(t)dt
\]

We are now in a position to determine \( \lambda \). We choose it so the coefficient of \( z^2 \) is zero. This gives a quadratic polynomial for \( \lambda \):

\[
-\lambda^2 + \lambda - \lambda(a+b+1) - ab = 0
\]
or
\[ \lambda^2 + \lambda(a + b) + ab = 0 \]
which has roots
\[ \lambda = -a; \quad \lambda = -b. \]
We can choose either root - each choice is associated with a different integral representation. In what follows we make the choice \( \lambda = -a \) which leads to the equations
\[ 0 = \int_C (z(a^2 + a - ca - ta(a + b + 1) + 2abt) + (ca - abt^2)) \lambda(z-t)^{-a-2}v(t)dt \]
\[ \int_C (z(a^2 + a - ca - ta(a - b + 1)) + (ca - abt^2)) (z-t)^{-a-2}v(t)dt \]
Next we replace \( z = z - t + t \) to get
\[ \int_C ((z - t)(a^2 + a - ca - ta(a - b + 1)) + t(a^2 + a - ta(a + 1)) (z-t)^{-a-2}v(t)dt = \]
\[ \int_C ((z - t)a(a + 1 - c - t(a - b + 1)) + t(1 - t)a(a + 1)) (z-t)^{-a-2}v(t)dt = \]
\[ \int_C ((z - t)a(-a + 1 - c + t(a - b + 1)) - t(1 - t)a(-a - 1)) (z-t)^{-a-2}v(t)dt. \]
This can be expressed as a differential operator in \( T \)
\[ 0 = \int_C ((a + 1 - c + t(b - a - 1)) \frac{d}{dt} + t(1 - t) \frac{d^2}{dt^2} (z-t)^{-a}v(t)dt. \]
We define the differential operator
\[ M_t = t(1 - t) \frac{d^2}{dt^2} + (a + 1 - c + t(b - a - 1)) \frac{d}{dt} \]
Following what was discussed in topic 2 of lecture 28 we want \( v(t) \) to be a solution of the adjoint equation which has the form:
\[ 0 = M^\dagger v(t) = \frac{d^2}{dt^2}t(1 - t)v(t) - \frac{d}{dt} (a + 1 - c + t(b - a - 1))v(t) = 0. \]
Integrating once gives
\[ \frac{d}{dt} t(1 - t)v(t) - ((a + 1 - c + t(b - a - 1))v(t) = 0 \]
To integrate this let

\[ w(t) := t(t-1)v(t) \]

which gives

\[ \frac{w'(t)}{w(t)} = -\frac{(a+1-c+t(b-a-1))}{t(t-1)} \]

Integrating once again gives

\[ \ln w(t) = -\int (a+1-c)(-\frac{1}{t} + \frac{1}{t-1}) - \int (b-a-1)(t-1)^{-1} \]

\[ \ln w(t) = (a+1-c)\ln(t) + (-1 - a + c - b + a + 1)\ln(t-1) \]

\[ w(t) = Ct^{a+1-c}(t-1)^{c-b} \]

\[ v(t) = Ct^{a-c}(t-1)^{c-b-1} \]

where \( C \) is a constant.

Next we use the Lagrange identity to determine the boundary conditions that make the surface term vanish. To do this consider the difference

\[ v(t)M_{t}(z-t)^{-a} - (z-t)^{-a}M_{t}^{1}v(t) \]

using the expressions for \( v(t) \) after some algebra gives the following expression for the boundary term:

\[ v(t)M_{t}(z-t)^{-a} - (z-t)^{-a}M_{t}^{1}v(t) = \frac{d}{dt}aCt^{a-c+1}(t-1)^{c-b}(z-t)^{-a-1}. \]

This vanishes for \( t = 1 \) and \( t = \infty \) provided

\[ \text{Re}(c) > \text{Re}(b) > 0 \]

The next step is to choose a contour between the two points. One choice is to directly integrate along the real axis from 1 to \( \infty \). Putting everything together we get the integral representation of the hypergeometric function

\[ F(a, b, c; z) = C \int_{1}^{\infty} dt(t-z)^{-a}t^{a-c}(t-1)^{c-b-1} \]

It remains to find the multiplicative constant \( C \). Here we changed the constant to express this in terms of \( t-z \) rather than \( z-t \). This is because \( t > 1 \) and we want to compare to the power series expansion of \( F(a, b, c; z) \) for \( |z| < 1 \)

Expanding \((t-z)^{-a}\) in powers of \( z \) using

\[ (t-z)^{-a} = t^{-a} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} \left(\frac{z}{t}\right)^{n} \]

and using this expansion in the integral representation gives

\[ F(a, b, c; z) = C \int_{1}^{\infty} dt(z-t)^{-a}t^{a-c}(t-1)^{c-b-1} = \]
\[ C \sum_{n=0}^{\infty} z^n \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} \int_{1}^{\infty} dt \, t^{-c-n}(t-1)^{c-b-1} \]

where the integral is the beta function (see page 96 equation 32.8 of the text)

\[ = C \sum_{n=0}^{\infty} z^n \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} = \]

\[ C \frac{\Gamma(c-b)\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} z^n \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(n+1)} \]

where this has the form of a constant times the series expansion of \( F(a, b, x; z) \):

\[ = C \frac{\Gamma(c-b)\Gamma(b)}{\Gamma(c)} F(a, b, c; z) \]

This allows us to calculate the coefficient \( C \):

\[ C = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \]

\[ F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{1}^{\infty} dt \, (z-t)^{-a} t^{a-c} (t-1)^{c-b-1} \]

For homework I will ask you to make the variable \( t \to 1/t \) in the integral representation to show \( F(a, b, c; z) \) can also be represented by

\[ F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{0}^{1} dt \, (1-zt)^{-a} t^{b-1} (1-t)^{c-b-1} \]

which converges for \( \text{Re}(c) > \text{Re}(b) > 0 \).