Lecture 29 - Topic 3: Relations

There are a number of identities associated with hypergeometric functions. If we start with the series representation of the hypergeometric function

\[ F(a, b, c; z) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} z^n. \]  

(1)

and differentiate \( n \) times we get

\[ \frac{d^n F(a, b, c; z)}{dx^n} := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+m+n-m)\Gamma(b+m+n-m)}{\Gamma(c+m+n-m)(n-m)!} z^{n-m} = \]

\[ \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(c+m)} F(a+m, b+m, c+m; z) \]

The functions \( F(a \pm 1, b, c; z) \), \( F(a, b \pm 1, c; z) \) and \( F(a, b, c \pm 1; z) \) are called hypergeometric contiguous to \( F(a, b, c; z) \). There are a number of relations between these functions that can be derived by comparing coefficients of the power series. Two examples are (1):

\[ (c-2a-(b-a)z)F(a, b, c; z) + a(1-z)F(a+1, b, c; z) - (c-a)F(a-1, b, c; z) = 0 \]

\[ (c-a-1)F(a, b, c; z) + aF(a+1, b, c; z) - (c-1)F(a, b+1; c; z) = 0 \]

We illustrate these last. Expressed in term of the series it has the form

\[ (c-a-1) \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} z^n + a \frac{\Gamma(c)}{\Gamma(a+1)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+1+n)\Gamma(b+n)}{\Gamma(c+n)n!} z^n - \]

\[ (c-1) \frac{\Gamma(c-1)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n-1)n!} z^n \]

The coefficient of \( z^n \) is

\[ \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \left( (c-a-1) \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} + a \frac{\Gamma(a+1+n)\Gamma(b+n)}{\Gamma(c+n)n!} - (c-1) \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n-1)n!} \right) = \]

\[ \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} \left( (c-a-1) + \frac{a}{a}(a+n) - \frac{c-1}{c-1}(c+n-1) \right) = 0 \]

There are a total of 15 such relations involving contiguous functions.

Another identity uses the integral representation mentioned at the end of the last topic.

\[ F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt (1-zt)^{-a} t^{b-1} (1-t)^{c-b-1} \]
Change variables so $t' = 1 - t$ gives

$$
= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt (1 - z + zt)^{-a} (1 - t)^{b-1} (1 - t)^{c-b-1}
$$

$$
\frac{\Gamma(c)(1 - z)^{-a}}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt \left(1 + \frac{zt}{1 - z}\right)^{-a} (1 - t)^{b-1} (1 - t)^{c-b-1} =
$$

$$(1 - z)^{-a} F(a, c - b, c, \frac{z}{z-1})
$$

These analytic functions agree when $|z| < |z-1| < 1$ and $c$ is not 0 or a negative integer. This integral representation extend the domain of analyticity to a much larger region.

Using the representation for the beta function again we get

$$
F(a, b, c; 1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt (1 - t)^{-a} t^{b-1} (1 - t)^{c-b-1} =
$$

$$
\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt t^{b-1} (1 - t)^{c-b-1-a} =
$$

$$
\frac{\Gamma(c)\Gamma(b)\Gamma(c-b-a)}{\Gamma(b)\Gamma(c-b)\Gamma(c-a)} = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)}
$$

We have determined how to express the independent solutions in the neighborhood of $z = 1$

$$
G_{11}(z) = F(a, b, a+b+1-c; 1-z)
$$

$$
G_{12}(z) = (1-z)^{c-a-b} F(c-b, c-a, 1+c-a-b; 1-z)
$$

In the region where the domains overlap, $F(a, b, c; z)$ should be a linear combination of these two functions

$$
F(a, b, c; z) = \alpha F(a, b, a+b+1-c; 1-z) + \beta (1-z)^{c-a-b} F(c-b, c-a, 1+c-a-b; 1-z)
$$

The coefficients can be found by setting $z = 0$ and $z = 1$. This gives the relations:

$$
F(a, b, c; 1) = \alpha F(a, b, a+b+1-c; 0) +
$$

$$
F(a, b, c; 0) = \alpha F(a, b, a+b+1-c; 1) + \beta F(c-b, c-a, 1+c-a-b; 1)
$$

Using

$$
F(a, b, c; 0) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}
$$

and

$$
F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)}
$$

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in the above equation and solving for $\alpha$ and $\beta$ gives

$$\alpha = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \quad \beta = \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}$$

which gives

$$F(a, b, c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b, a+b+1-c; 1-z) + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} F(c-b, c-a, 1+c-a-b; 1-z)$$

This provides an analytic continuation of $F(a, b, c; z)$ in a neighborhood of $z = 1$. There are many more relations that can be derived among the solutions of the hypergeometric differential equation.