Lecture 29 - Topic 4: Relation of special functions

Some of the equations that govern the classical orthogonal polynomials are the Jacobi differential equation
\[(1-z)^2 \frac{d^2 u(z)}{dz^2} + (\beta - \alpha(\alpha + \beta + 2)z) \frac{du(z)}{dz} + \lambda(\alpha + \beta + 1)u(z) = 0\]
and the Gegenbauer equation
\[(1-z)^2 \frac{d^2 u(z)}{dz^2} + (2\mu + 1)z \frac{du(z)}{dz} + \lambda(\lambda + 2\mu)u(z) = 0.\]

Making the substitution \(z = 1 - 2x = 1 - 2z'\) transforms these to equations of the hypergeometric type
\[z(1-z)u''(z) + [c - z(a + b + 1)]u'(z) - abu(z) = 0.\]

This is called the Jacobi function of the first kind.

\[
P^{(\alpha, \beta)}(z) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)\Gamma(n + \alpha + \beta + 1)} F(-\lambda, \lambda + \alpha + \beta + 1, \alpha + 1, 1 - \frac{z}{2})
\]

Both of these solutions become polynomials when \(\lambda = n \geq 0\) is a non-negative integer. To derive this, we take the limit \(\lambda \to \mu\) in the series solution. For example, consider

\[
\lim_{\lambda \to \mu} G^{(\mu)}(z) = \lim_{\lambda \to \mu} \frac{\Gamma(2\mu + \lambda)}{\Gamma(\lambda + 1)\Gamma(2\mu)} F(-\lambda, \lambda + \mu + 1, \frac{1}{2}, 1 - \frac{z}{2})
\]

Properties of the Gamma function give the factor

\[
\lim_{\lambda \to \mu} \frac{\Gamma(-\lambda + n)}{\Gamma(-\lambda)} = \left\{ \begin{array}{cl} (-)^n n! & n \leq m \\ 0 & n > m \end{array} \right.
\]

eliminates the terms for \(n > m\). The same mechanism makes the Jacobi function of the first kind become a polynomial when \(\lambda = n \geq 0\).

These are the analytic solutions of these equations - the other independent solutions of these equations are not analytic. They are also related to the hypergeometric functions by

\[
Q^{(\alpha, \beta)}(z) = \frac{2^{\lambda + \alpha + \beta} \Gamma(\lambda + \alpha + 1)\Gamma(\lambda + \beta + 1)}{\Gamma(2\lambda + \alpha + \beta + 2)(z - 1)^{\lambda + \alpha + 1}(z + 1)^{\beta}} F(\lambda + 1, \lambda + \alpha + 1, 2\lambda + \alpha + \beta + 2, 1 - z).
\]
This solution is called the Jacobi function of the second kind. There is also a
not analytic solution independent solution of the Gegenbauer equations. It is
proportional to a particular Jacobi function of the second kind:

$$Q_{\lambda}^{\mu,\frac{1}{2},\frac{1}{2}}(z)$$

The most important special case of the Jacobi functions are the case $\alpha = \beta = 0$ which give the Legendre polynomials. The Legendre function of the first
kind is

$$P_{\lambda}(z) = P_{\lambda}^{(0,0)}(z)$$

which becomes a polynomial when $\lambda = n$ is a non-negative integer.

The second independent non-analytic solution is called the Legendre function
of the second kind. It is related to the Jacobi function of the second kind by

$$Q_{\lambda}(z) = Q_{\lambda}^{(0,0)}(z)$$

The important observation is how all these solutions re related to the hyper-
geometric function, $F(a, b, c; z)$