Lecture 30 - Topic 1: The confluent hypergeometric function

We start by considering the special case of the Riemann equation:

$$\frac{d^2 u(z)}{dz^2} + \left( \frac{c}{z} + \frac{1 - a - b}{z - z_2} + \frac{1 - c - a + b}{z - z_3} \right) \frac{du(z)}{dz} + \frac{abz_2(z_2 - z_3)}{z(z - z_2)(z - z_3)} u(z) = 0$$

This has regular singular points at $z = 0, z_2$ and $z_3$. Let $z_2 = 2z_3 = b$ and take the limit as $b \to \infty$, keeping $a$ and $c$ fixed.

In this limit the Riemann equation reduces to

$$\frac{d^2 u(z)}{dz^2} + \left( \frac{c}{z} + 1 - 2 \right) \frac{du(z)}{dz} - \frac{au(z)}{z} = 0$$

Multiplying by $z$ gives

$$z \frac{d^2 u(z)}{dz^2} + (c - z) \frac{du(z)}{dz} - au(z) = 0.$$  

This is called the confluent Riemann equation. This is a limiting form of the hypergeometric equation where the singularity at 1 is moved out to infinity. In this limit $z = \infty$ is no longer a regular singular point. To see this let $w = 1/z$, then

$$\frac{d}{dz} = \frac{dw}{dz} \frac{d}{dw} = -\frac{1}{z^2} \frac{d}{dw} = -w^2 \frac{d}{dw}$$

and

$$\frac{d^2}{dz^2} = \left( w^2 \frac{d}{dw} \right)^2 \frac{d}{dw} = w^4 \frac{d^2}{dw^2} + 2w^3 \frac{d}{dw}.$$  

Using these in the differential equation gives

$$\frac{1}{w} \left( w^4 \frac{d^2 u}{dw^2} + 2w^3 \frac{du}{dw} \right) - w^2 \frac{c}{w} \frac{du}{dw} - au = 0$$

Dividing by $w^3$ gives

$$\frac{d^2 u}{dw^2} + \left( \frac{2 - c}{w} + \frac{1}{w^2} \right) \frac{du}{dw} - \frac{a}{w^3} u = 0$$

which shows that the coefficients of both $u$ and $\frac{du}{dw}$ are too singular at the origin, $(w = 0 \to z = \infty)$, to be considered regular.

The indicial equation for the confluent Riemann equation equation is

$$0 = r(r - 1) + cr - r(r + c - 1)$$

which has roots $r = 0$ and $r = 1 - c$. The solution that is analytic in the neighborhood of the origin is denoted by

$$\Phi(a, c; z)$$

with normalization

$$\Phi(a, c; 0) = 1.$$
Since the only other singularities in the equation are at $\infty \Phi(a, c; z)$ is an entire function called the confluent hypergeometric function.

We will construct this solution using the method of integral representations again. In this case we use the Laplace kernel and write

$$u(z) = \int_C e^{zt} v(t) dt.$$ 

The first step is to determine the differential operator $M_t$.

$$L_z u(z) = \int_C (zt^2 + (c - z)t - a) e^{zt} v(t) dt.$$ 

Next we replace the $z$ dependence by a differential operator in $t$:

$$L_z u(z) = \int_C v(t) \left( t^2 \frac{d}{dt} + (c - t) - a \right) e^{zt} dt = \int_C v(t) \left( t^2 - t \frac{d}{dt} + (tc - a) \right) e^{zt} dt.$$ 

Integrating by parts gives the adjoint operator

$$\int_C e^{zt} \left( -\frac{d}{dt}(t^2 - t) + (tc - a) \right) v(t) dt$$

where we have to choose $C$ so there are no boundary terms. To do this we first have to find solutions $v(t)$ of the adjoint equation. To find homogeneous solutions to the adjoint equation we integrate

$$-(t^2 - t)v' - (2t - 1 + a - ct)v = 0$$

which can be expressed as

$$\frac{v'}{v} = -\frac{(2 - c)}{t - 1} + (1 - a)(-\frac{1}{t} + \frac{1}{t - 1}).$$

Integrating

$$ln(v) = ln(t - 1)^{c-1-a} + ln(t)^{a-1}$$

$$v = C(t - 1)^{c-1-a}t^{a-1}$$

where $C$ is a constant. The boundary terms that must vanish when integrating by parts are

$$e^{zt}(t^2 - t)v(t) = e^{zt}(t^2 - t)C(t - 1)^{c-1-a}t^{a-1} = Ce^{zt}(t - 1)^{c-a}t^a.$$ 

The path of integration must be chosen such that either vanishes at the endpoints or have identical values at the endpoints. If $Re(c) > Re(a) > 0$ then this vanishes at $t = 0$ and $t = 1$. This results in the integral representation

$$u(z) = C \int_0^1 e^{zt}t^{a-1}(1 - t)^{c-a-1} dt.$$
where we have changed $C$ to change $(t - 1)$ to $(1 - t)$. To find the normalization condition we expand the exponential term in powers of $z$:

$$u(z) = C \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_0^1 t^{a-1+n} (1 - t)^{c-a-1} dt$$

where the integral is $B(a + n, c - a)$ which gives

$$u(z) = C \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma(a + n) \Gamma(c - a)}{\Gamma(n + 1) \Gamma(c + n)}$$

$$1 = u(0) = C \frac{\Gamma(a) \Gamma(c - a)}{\Gamma(c)}$$

or

$$C = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c - a)}.$$

This gives the integral representation and the series representation of

$$\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c - a)} \int_0^1 e^{zt} t^{a-1} (1 - t)^{c-a-1} dt =$$

$$\frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)}{\Gamma(n + 1) \Gamma(c + n)} z^n \frac{\Gamma(a + n) \Gamma(b + n)}{\Gamma(n + 1) b^n \Gamma(c + n)}.$$

Comparing the expansion with the corresponding expansion for the the hypergeometric functions

$$\lim_{b \to \infty} F(a, b, c, \frac{z}{b}) = \lim_{b \to \infty} \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n) \Gamma(b + n)}{\Gamma(n + 1) b^n \Gamma(c + n)} =$$

$$\frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)}{\Gamma(n + 1) \Gamma(c + n)} = \Phi(a, c; z)$$

where we have used

$$1 = \lim_{b \to \infty} \frac{\Gamma(b + n)}{b^n \Gamma(b)} = \lim_{b \to \infty} (1 + \frac{n-1}{b})(1 + \frac{n-2}{b}) \cdots (1 + \frac{1}{b})1.$$

This gives the relation of the confluent hypergeometric function to the hypergeometric function:

$$\Phi(a, c; z) = \lim_{b \to \infty} F(a, b, c, \frac{z}{b}).$$

We get another integral representation of $\Phi(a, c; z)$ using the above limit in the Euler integral representation of the hypergeometric equation

$$\Phi(a, c; z) = \lim_{b \to \infty} F(a, b, c, \frac{z}{b}) = \lim_{b \to \infty} F(b, a, c, \frac{z}{b}) =$$
\[
\lim_{b \to \infty} \frac{\Gamma(c)}{\Gamma(a) \Gamma(c - a)} \int_0^1 dt (1 - \frac{t z}{b})^{-b} t^{a-1} (1 - t)^{c-a-1} - b t^{a-1} (1 - t)^{c-a-1}
\]
where we have used the symmetry of hypergeometric function on interchanging \(a\) and \(b\) and
\[
\lim_{b \to \infty} (1 - \frac{t z}{b})^{-b} = e^{zt}
\]
which can be understood by writing
\[
\lim_{b \to \infty} (1 - \frac{t z}{b})^{-b} = (1 + \frac{t z}{b})^b = e^{zt}.
\]
This integral representation is valid for \(\text{Re}(c) > \text{Re}(a) > 0\).

There is also a second non-analytic solution to the hypergeometric differential equation.

Recall when \(c\) is not an integer the non-analytic solution of the hypergeometric equation in a neighborhood of 0 is
\[
\mathcal{Z}^{1-c}F(b - c + 1, a - c + 1, 2 - c; z)
\]
The second independent solution of the hypergeometric equation is obtained by taking the same limit of this equation
\[
\lim_{b \to \infty} \mathcal{Z}^{1-c}F(b - c + 1, a - c + 1, 2 - c; \frac{z}{b}) = \mathcal{Z}^{1-c}F(a - c + 1, b - c + 1, 2 - c; z) = \mathcal{Z}^{1-c}\Phi(a - c + 1, 2 - c, z).
\]
There is another independent solution. Previously we derived the integral representation
\[
u(z) = C \int_c e^{zt} t^{a-1} (1 - t)^{c-a-1} dt\]
with boundary term
\[
C e^{zt} (t - 1)^{c-a}.
\]
This vanishes at 0 and \(-\infty\) is \(\text{Re}(a) > 0\) and \(\text{Re}(z) > 0\). In this case we get the integral representation
\[
u(z) = C \int_{-\infty}^0 e^{zt} t^{a-1} (1 - t)^{c-a-1} dt.
\]
Changing \(t \to -t\) gives
\[
\Phi(a, c; z) := \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1 + t)^{c-a-1} dt.
\]
This solution can be expressed as a linear combination of
\[
\Phi(a, c; z) = \alpha \Phi(a, c; z) + \beta \mathcal{Z}^{1-c}\Phi(a - c + 1, 2 - c, z).
\]
The coefficients can be computed by comparing the expressions at 2 values. The result is

\[ \Phi(a, c; z) = \frac{\Gamma(c - 1)}{\Gamma(a - c + 1)} \Phi(a, c; z) + \frac{\Gamma(c - 1)}{\Gamma(a)} z^{1-c} \Phi(a - c + 1, 2 - c, z) \]