Lecture 30 - Topic 2: Functions related to the confluent hypergeometric function

Parabolic cylinder functions, Hermite functions, Laguerre functions

One of the reasons for studying the confluent hypergeometric functions is because it is related to so many of the standard functions that appear in theoretical physics.

The first class are solutions of the following differential equation

\[
\frac{d^2 u(z)}{dz^2} + \left( \nu + \frac{1}{2} - \frac{z^2}{4} \right) u(z) = 0.
\]

This is called the Weber-Hermite differential equation. We make the substitution

\[ u(z) = e^{-\frac{1}{4}z^2} \tilde{u}(z). \]

This gives

\[
\begin{align*}
    u'(z) &= e^{-\frac{1}{4}z^2} \left( \tilde{u}'(z) - \frac{z}{2} \tilde{u}(z) \right), \\
    u''(z) &= e^{-\frac{1}{4}z^2} \left( \tilde{u}''(z) - \frac{1}{2} \tilde{u}(z) - \frac{z}{2} \tilde{u}'(z) - \frac{z^2}{4} \tilde{u}'(z) + \frac{z^2}{4} \tilde{u}(z) \right) = \\
    u''(z) &= e^{-\frac{1}{4}z^2} \left( \tilde{u}''(z) - \frac{1}{2} \tilde{u}(z) - z \tilde{u}'(z) + \frac{z^2}{4} \tilde{u}(z) \right).
\end{align*}
\]

Using these expressions in the differential equation leads to the following differential equation for \( \tilde{u}(z) \):

\[
0 = \tilde{u}''(z) - \frac{1}{2} \tilde{u}(z) - z \tilde{u}'(z) + \frac{z^2}{4} \tilde{u}(z) + (\nu + \frac{1}{2} - \frac{z^2}{4}) \tilde{u}(z) = \\
\tilde{u}''(z) - z \tilde{u}'(z) + \nu \tilde{u}(z) = 0.
\]

The next step is to relate this to the solution of the confluent hypergeometric equation. It turns out that \( \tilde{u}(z) \) is related to \( u(z^2/2) \) where \( u(z^2/2) \) is a confluent hypergeometric function.

Note that

\[
\frac{du(z^2/2)}{dz} = zu'(z^2/2)
\]

\[
\frac{d^2 u(z^2/2)}{dz^2} = u'(z^2/2) + z^2 u''(z^2/2)
\]

using these expression in the equation

\[
\tilde{u}''(z) - z \tilde{u}'(z) + \nu \tilde{u}(z) = 0
\]

gives

\[
z^2/2u''(z^2/2) + (1/2)u'(z^2/2) - z^2/2u'(z^2/2) + \nu/2u(z^2/2).
\]

Comparing this to the confluent hypergeometric differential equation

\[
z \tilde{u}''(z) + (c - z)u'(z) - au(z) = 0
\]
we find that \( \tilde{u}(z) \) is a solution to the hypergeometric equation with \( c = 1/2 \) and \( a = -\nu/2 \).

Combining everything together gives the following solution to the Weber-Hermite differential equation, which is called the Parabolic Cylinder function:

\[
D_{\nu}(z) = 2^{\frac{\nu}{2}} e^{-z^2/4} \Psi\left(-\frac{\nu}{2}, \frac{1}{2}; \frac{z^2}{2}\right)
\]

where the \( 2^{\frac{\nu}{2}} \) is a standard normalization. Here \( \Psi(a, c; z) \) is the second solution of the confluent hypergeometric equation.

It can be expressed in terms of the confluent hypergeometric function using the identity from the last topic:

\[
\Psi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c)} \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma(a + n)}{\Gamma(n + 1) \Gamma(n + c)}
\]

when \( a = -m \) is a negative integer. This is because \( \Gamma(-m) \) is infinite. To see this note

\[
\Phi(-m, c; z) = \frac{\Gamma(c)}{\Gamma(-m)} \sum_{n=0}^{m} (-1)^n \frac{z^n}{n!} \frac{\Gamma(n - m)}{\Gamma(n + 1) \Gamma(n + c + m)}
\]

which means that \( \Phi(-m, c; z) \) is a degree \( m \) polynomial.

If we apply this to

\[
\Psi\left(-\frac{\nu}{2}, \frac{1}{2}; \frac{z^2}{2}\right)
\]

when \( n \) is a non-negative integer then using (1) for integer \( \nu = m \), if \( \nu = 2m \) is even then

\[
\frac{1}{\Gamma\left(-\frac{2m}{2}\right)} = \frac{1}{\Gamma(-m)} = 0
\]

and

\[
\Phi\left(-\frac{2m}{2}, \frac{1}{2}; \frac{z^2}{2}\right) \Phi(-m, \frac{1}{2}; \frac{z^2}{2})
\]
are polynomials, while if \( \nu = 2m + 1 \) is odd
\[
\frac{1}{\Gamma\left(1-2m-\frac{1}{2}\right)} = \frac{1}{\Gamma(-m)} = 0
\]
and
\[
\Phi\left(\frac{1 - 2m - 1}{2}, \frac{3}{2}, \frac{z^2}{2}\right) \Phi\left(-m, \frac{3}{2}, \frac{z^2}{2}\right)
\]
are polynomials. In both cases
\[
\Psi\left(-\frac{m}{2}, \frac{1}{2}, \frac{z^2}{2}\right)
\]
is a polynomial of degree \( m \).

The polynomials
\[
H_n(z) = 2^n \Psi\left(-\frac{n}{2}, \frac{1}{2}, \frac{z^2}{2}\right) = 2^n e^{z^2/4} D_n(z)
\]
are the Hermite polynomials that we encountered in the context of studying classical orthogonal polynomials.

We showed above that the confluent hypergeometric function for negative integer \( a \) are also polynomials. They are up to a multiplicative constant the associated Laguerre polynomials
\[
L_n^\mu(z) = \frac{\Gamma(n + \mu + 1)}{\Gamma(n + 1)\Gamma(\mu + 1)} \Phi(-n, \mu + 1, z)
\]
These functions appear in eigenfunctions of the Hydrogen atom Hamiltonian.