Bessel’s equation is
\[ b''(z) + \frac{1}{z} b'(z) + \left(1 - \frac{\nu^2}{z^2}\right) b(z) = 0. \]

To show how Bessel’s equation is related to the confluent hypergeometric equation define \( u(z) \) by let
\[ b(z) = z^\nu e^{-iz} u(z) \]

and compute
\[ b'(z) = z^\nu e^{-iz} (u'(z) + \frac{\nu}{z} u(z) - i u(z)) \]
\[ b''(z) = z^\nu e^{-iz} \left( u''(z) + \frac{\nu}{z} u'(z) - \frac{\nu^2}{z^2} u(z) + (u'(z) + \frac{\nu}{z} u(z) - i u(z)) \left( \frac{\nu}{z} - i \right) \right) = \]
\[ z^\nu e^{-iz} \left( u''(z) + \frac{\nu}{z} u'(z) - i u'(z) - \frac{\nu}{z^2} u(z) + \frac{\nu}{z} u'(z) - i u'(z) - \frac{i \nu}{z} u(z) + \left( \frac{\nu}{z} \right)^2 u(z) - u(z) - \frac{i \nu}{z} u(z) \right) \]
\[ z^\nu e^{-iz} \left( u''(z) + 2 \frac{\nu}{z} u'(z) - 2 i u'(z) - \frac{\nu}{z}^2 u(z) - \frac{2 i \nu}{z} u(z) + \left( \frac{\nu}{z} \right)^2 u(z) - u(z) \right) . \]

With these substitutions Bessel’s equation becomes
\[ u'(z) + 2 \frac{\nu}{z} u'(z) - 2 i u'(z) - \frac{\nu}{z^2} u(z) - \frac{2 i \nu}{z} u(z) + \left( \frac{\nu}{z} \right)^2 u(z) - u(z) + \frac{2 i \nu}{z} u(z) - \frac{\nu}{z}^2 u(z) = 0 \]
\[ u'(z) + (\frac{2 \nu + 1}{z} - 2 i) u'(z) + (- \frac{\nu}{z^2} - \frac{2 i \nu}{z} + \left( \frac{\nu}{z} \right)^2 - 1 + \frac{\nu}{z} - \frac{i \nu}{z} - \frac{\nu}{z^2} ) u(z) = \]
\[ u'(z) + (\frac{2 \nu + 1}{z} - 2 i) u'(z) + (- \frac{2 i \nu}{z} - \frac{i}{z} ) u(z) . \]

Multiplying the above equation by \( z \) gives
\[ z u''(z) + (2 \nu + 1 - 2 i z) u'(z) - i (2 \nu + 1) u(z) = 0 . \]

A solution of this equation is given by the confluent hypergeometric function
\[ \Phi(\nu + \frac{1}{2}, 2 \nu + 1; 2 i z) \]
expressing \( b(z) \) in terms of this \( u(z) \) gives a solution to Bessel’s equation
\[ b(z) = z^\nu e^{-iz} \Phi(\nu + \frac{1}{2}, 2 \nu + 1; 2 i z) \]
or
\[ J_\nu(z) := \frac{1}{\Gamma(\nu + 1)} \left( \frac{z}{2} \right)^\nu e^{-iz} \Phi(\nu + \frac{1}{2}, 2 \nu + 1; 2 i z) \]

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which differs from \( b(x) \) by a multiplicative constant. \( J_\nu(z) \) is called the Bessel function of the first kind of order \( \nu \).

Using the series solution of the confluent hypergeometric equation

\[
\Phi(a, c, z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)\Gamma(n+1)} z^n
\]

and

\[
e^{-iz} = \sum_{m=0}^{\infty} \frac{(-iz)^m}{m!}
\]

in the definition of \( J_\nu(z) \) gives

\[
J_\nu(z) := \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \sum_{m,n=0}^{\infty} \frac{(-iz)^m}{m!} \frac{\Gamma(2\nu+1)}{\Gamma(\nu+\frac{3}{2})} \frac{\Gamma(\nu+n+\frac{1}{2})}{\Gamma(2\nu+n+1)\Gamma(n+1)} (2iz)^n = \\
\frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \frac{i^n(-2)^m n! \Gamma(\nu+m+\frac{1}{2})\Gamma(2\nu+1)}{m!(b-m)! \Gamma(\nu+\frac{3}{2})\Gamma(2\nu+m+1)} \right) \frac{z^n}{n!} = \\
\left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1+\nu+n)} \left( \frac{z}{2}\right)^{2n}
\]

(I was able to verify that this is correct up to order \( z^4 \) by computing the coefficients of powers of \( z \) but was unable to express the series solution in the form directly above, however this is a well-known relation).

For the confluent hypergeometric function the second solution corresponding to \( \Phi(a, c; z) \) is \( z^{1-c} \Phi(a - c + 1, 2 - c, z) \). Letting \( a = \nu + \frac{1}{2} \) and \( c = 2\nu + 1 \) the second solution has the form

\[
z^{-2\nu} \Phi\left(\frac{1}{2} - \nu, 1 - 2\nu; z\right)
\]

which leads to a second solution of Bessel’s equation

\[
\left(\frac{z}{2}\right)^\nu e^{-iz} z^{-2\nu} \Phi\left(\frac{1}{2} - \nu, 1 - 2\nu; 2iz\right) = \\
\left(\frac{1}{2}\right)^{2\nu} \left(\frac{z}{2}\right)^{-\nu} e^{-iz} \Phi\left(\frac{1}{2} - \nu, 1 - 2\nu; 2iz\right).
\]

By inspection we can see that this function is proportional to \( J_{-\nu}(z) \), which shows that the independent solutions of Bessel’s equation are \( J_\nu(z) \) and \( J_{-\nu}(z) \).

This is true when \( \nu \) is not an integer. When \( \nu \) is an integer then these two solutions become independent. This can be seen by comparing the series solutions. First note that in the series expression for \( J_{-n}(z) \):

\[
\left(\frac{z}{2}\right)^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(1-n+m)} \left( \frac{z}{2}\right)^{2m}
\]
the factor $\Gamma(1-n+m)$ in the denominator is infinite for $m < n - 1$ so the sum starts at $n$ giving
\[
= \left(\frac{z}{2}\right)^{-n} \sum_{m=n}^{\infty} \frac{(-1)^m}{m! \Gamma(1-n+m)} \left(\frac{z}{2}\right)^{2m}.
\]
Let $k = m - n$ gives
\[
\left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{(k+n)! (1+k)} \left(\frac{z}{2}\right)^{2(k+n)} = \frac{(-1)^n}{n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(1+k)} \left(\frac{z}{2}\right)^{2k} = (-)^n J_n(z)
\]
Which shows
\[
J_{-n}(z) = (-)^n J_n(z).
\]
Up to the factor $(-)^n$ this series is identical to the series for $J_n(z)$. In this case to construct a second independent solution first define
\[
Y_\nu(z) := \frac{J_\nu(z) \cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}
\]
which is a linear combination of both solutions when $\nu \neq 0$. $Y_\nu(z)$ is called the Bessel function of the second kind or Neumann function. $J_\nu(z)$ and $Y_\nu(z)$ are also a set of independent solutions of Bessel’s equation.

As $\nu \to n$ where $n$ is an integer both the numerator and denominator of the expression for $Y_\nu(z)$ vanish, but there is a well defined limit
\[
Y_n(z) := \lim_{\nu \to n} Y_\nu(z) = \lim_{\nu \to n} \frac{1}{\pi} \left\{ \frac{\partial J_\nu(z)}{\partial \nu} - (-)^n \frac{\partial J_{-\nu}(z)}{\partial \nu} \right\}
\]
which is an independent solution of Bessel’s equation. This can be seen by applying Bessel’s equation to the expression for $Y_n(z)$
\[
L_x Y_n(z) = \lim_{\nu \to n} \frac{1}{\pi} \left\{ L_x \frac{\partial J_\nu(z)}{\partial \nu} - (-)^n L_x \frac{\partial J_{-\nu}(z)}{\partial \nu} \right\}
\]
\[
\lim_{\nu \to n} \frac{1}{\pi} \left\{ \frac{\partial L_x J_\nu(z)}{\partial \nu} - (-)^n \frac{\partial L_x J_{-\nu}(z)}{\partial \nu} \right\} + \lim_{\nu \to n} \frac{1}{\pi} \left\{ \frac{\partial L_x J_{-\nu}(z)}{\partial \nu} \right\} - (-)^n \frac{1}{\pi} (J_\nu(z) - (-)^n J_{-\nu}) 2\nu/z^2
\]
which vanishes because $L_x J_{-\nu}(z) = 0$ and $(J_n(z) - (-)^n J_{-n}) = 0$.

Any linear independent linear combinations of independent solutions are independent solutions. Two solutions that are important in applications are the following linear combinations:
\[
H_1^\nu(z) := J_\nu(z) + iY_\nu(z) \quad H_2^\nu(z) := J_\nu(z) - iY_\nu(z).
\]
These are called Bessel functions of the third kind or Hankel functions. The reason for considering different sets of independent solutions is that some choices are natural for satisfying boundary conditions for a particular problems.