Lecture 31 - Topic 1: Bessel functions of imaginary argument

Bessel functions

One of the properties of solutions of linear differential equations with coefficients that are analytic in a region is that the solutions are analytic the same region.

This means that the solutions of Bessel’s equation are analytic functions of $z$ in their domain of analyticity.

Substitution of $z \to iz$ in Bessel’s equation gives the equation

$$b''(z) + \frac{1}{z}b'(z) - \left(1 + \frac{\nu^2}{z^2}\right)b(z) = 0.$$ 

Independent solutions are $J_{\pm\nu}(iz)$ when $\nu$ is not an integer. Standard definitions are

$$I_{\nu}(z) = e^{-i\pi\nu}J_{\nu}(iz) = \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^{2n+\nu} \frac{1}{n!\Gamma(n+\nu+1)}$$

which is called the modified Bessel function of the first kind. The other independent solution usually taken as

$$K_{\nu}(z) := \frac{\pi}{2\sin(\nu\pi)}[I_{\nu}(z) - I_{-\nu}(z)]$$

which is called the modified Bessel function of the second kind.

When $\nu$ is an integer $K_{\nu}(z)$ is defined in a similar manner to $J_{\nu}(z)$:

$$K_{n}(z) = \lim_{\nu \to n} K_{\nu}(z) = \frac{(-)^n}{2} \lim_{\nu \to n} \left[\frac{\partial I_{\nu}(z)}{\partial \nu} - \frac{\partial I_{-\nu}(z)}{\partial \nu}\right]$$

The proof that this is a solution of the differential equation is identical to the corresponding proof for $J_{n}(z)$. 