The boundary conditions that appear in the Cauchy-Kovalevskaya theorem are of a specific type. In this section we consider a more general boundary conditions for a more restrictive class of equations.

Partial differential equations where the highest derivatives appear linearly are called quasi-linear equations. We further restrict out considerations to second order quasi-linear equations.

These have equations the form
\[
\sum_{m,n}^{D} a_{m,n}(x) \frac{\partial^2 u(x)}{\partial x_m \partial x_n} + F(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_D}) = 0.
\]

Since
\[
\frac{\partial^2 u(x)}{\partial x_m \partial x_n} = \frac{\partial^2 u(x)}{\partial x_n \partial x_m}
\]

we can assume without loss of generality that \(a_{m,n}(x) = a_{n,m}(x)\). If we also assume that the \(a_{m,n}(x)\) are real then it then follows that \(a_{mn}(x)\) is a real symmetric matrix for each \(x\). If we fix a value \(x_0\) of \(x\) the we can find a real \(D \times D\) orthogonal matrix \(O_{mn}\) that diagonalizes \(a_{mn}(x_0)\)

\[
\sum_{k=1}^{D} O_{km} O_{kn} = \delta_{mn}.
\]

where

\[
\sum_{k=1}^{D} O_{km} a_{mn}(x_0) O_{ln} = \delta_{kl} a_{l}(k_0).
\]

Define new variables
\[
y = O x \quad y_m = \sum_{n=1}^{D} O_{mn} x_n
\]

With this definition
\[
\frac{\partial y_m}{\partial x_m} = O_{mn}
\]

It follows that
\[
\sum_{m,m',n,n'}^{D} a_{m',n'}(x_0) \frac{\partial^2 u(x(y))}{\partial y_m \partial y_n} \frac{\partial y_m}{\partial x_m'} \frac{\partial y_n}{\partial x_n'} = \sum_{m,n,n'}^{D} O_{mm'} a_{m',n'}(x_0) O_{nn'} \frac{\partial^2 u(x(y))}{\partial y_m \partial y_n} = \sum_{m,n,n'}^{D} O_{mm'} a_{m',n'}(x_0) O_{nn'} \frac{\partial^2 u(x(y))}{\partial y_m \partial y_n} =
\]
\[
\sum_{m=1}^{D} a_m(x_0) \frac{\partial^2 u(x(y))}{\partial y_m^2} =
\]

With this variable change the equation at \( y = y_0 = y(x_0) \) becomes

\[
\sum_{m=1}^{D} a_m(x_0) \frac{\partial^2 u(x(y))}{\partial y_m^2} + F(y, u, \frac{\partial u}{\partial y_1}, \cdots, \frac{\partial u}{\partial y_D}) = 0. \tag{1}
\]

This structure is used to classify the types of differential equation at a point \( y = y_0 = y(x_0) \):

1. Equation (1) is **elliptic** at \( y = y_0 \) if all of the coefficients \( a_m(y_0) \) are non-zero and have the same sign.
   
   example:
   
   \[
   \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0.
   \]

2. Equation (1) is **hyperbolic** at \( y = y_0 \) if all of the coefficients \( a_m(y_0) \) are non-zero but do not all have the same sign.

   example:

   \[
   \frac{\partial^2 u(x, y)}{\partial x^2} - \frac{\partial^2 u(x, y)}{\partial y^2} = 0.
   \]

3. Equation (1) is **parabolic** at \( y = y_0 \) if at least one of the coefficients \( a_m(y_0) \) is zero.

   example:

   \[
   \frac{\partial^2 u(x, y)}{\partial x^2} - \frac{\partial u(x, y)}{\partial y} = 0.
   \]