Lecture 33 - Topic 2: Laplace’s equation

Laplace’s equation in two dimensions has the form

$$\frac{\partial u(x, y)}{\partial x^2} + \frac{\partial u(x, y)}{\partial y^2} = 0$$

In this case $A = C = 1$ and $B = 0$. In this case the characteristic curves are $y = \pm ix$ so there are no characteristic curves in the $x - y$ plane.

We assume that $u(x, y)$ is a solution of Laplace’s equation in a region bounded by a closed curve in the $x - y$ plane. First note that if there are two solutions with the same values on the curve the difference of the solutions also satisfies Laplace’s equation and is identically zero in the curve. We know that this is the real part of an analytic function that vanishes on the curve, so it must be zero. This shows that if a solution exists it must be unique. In general the existence is more complicated.

When $C$ is a circular boundary Poisson’s integral formula (from last semester) gives the solution

$$u(r \cos(\theta), r \sin(\theta)) = \frac{1}{2\pi} \int_0^{2\pi} d\theta' u(R, \theta') \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta') + r^2}.$$ 

In this case the Dirichlet boundary condition give a unique solution. It is clear that in this case the Cauchy boundary condition would lead to an over determined problem.

These examples suggest the following general considerations for choosing appropriate boundary conditions:

a Elliptic equations - use Dirichlet or Neumann boundary conditions on a closed hypersurface.

b Parabolic equations - use Dirichlet or Neumann boundary conditions on one side of an open hypersurface.

b Hyperbolic equations - use Cauchy conditions on an open hypersurface.