Lecture 33 - Topic 3: Multi-dimensional Fourier Transforms

The discussion of Fourier transform in one dimension extends to the many-dimensional case. We use the following notation

$$\int d^Nx = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_N.$$  

If \( f(x_1, \cdots, x_N) \) is a function of \( N \) variables the Fourier transform of \( f(x_1, \cdots, x_N) \) is defined by

$$F(k_1, \cdots, k_N) := \frac{1}{(2\pi)^{N/2}} \int d^Nxf(x_1, \cdots, x_N)e^{i\sum_{m=1}^{N} x_m k_m}$$

with the inverse transform defined by

$$f(x_1, \cdots, x_N) = \frac{1}{(2\pi)^{N/2}} \int d^NkF(k_1, \cdots, k_N)e^{-i\sum_{m=1}^{N} x_m k_m}.$$  

The proof follows from the one dimensional case because products of harmonic oscillator functions are a basis on an \( N \)-dimensional space. The multi-dimensional Fourier transform has many of the same properties as the one-dimensional Fourier transform:

$$f(x_1, \cdots, x_N) = \frac{1}{(2\pi)^{N}} \int d^Nk \int d^Nyf(y_1, \cdots, y_N)e^{i\sum_{m=1}^{N} (y_m - x_m)k_m}$$

which gives the Fourier representation of a delta function

$$\delta(x - y) = \prod_{n=1}^{N} \delta(x_n - y_n) = \frac{1}{(2\pi)^{N}} \int d^Nk e^{i\sum_{m=1}^{N} (y_m - x_m)k_m}.$$  

Like in the one dimensional case, this must be understood as a distributional relation. This means in a calculation the \( y \) integral over a suitable test function must be computed before the \( k \) integrals.

Delta functions behave very much like ordinary functions under variable changes. Consider a variable change \( x \to x' \). In the one-dimensional case

$$\int f(y) = \int f(x)\delta(x - y) = \int f(x(x'))\delta(x(x') - y(y'))\left|\frac{dx}{dx'}\right|dx' = \int f(x(x'))\delta(x' - y')dx'.$$

Comparing these expressions gives

$$\delta(x' - y') = \delta(x - y)\left|\frac{dx}{dx'}\right|.$$
The same algebra holds in the multidimensional case except \(|\frac{dx}{dx'}|\) is replaced by the absolute value of the Jacobian of the variable change

\[
J \left( \frac{\partial(x'_1, \cdots, x'_N)}{\partial(x_1, \cdots, x_N)} \right) = | \det \left( \begin{array}{ccc}
\frac{\partial x_1}{\partial x'_1} & \cdots & \frac{\partial x_1}{\partial x'_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial x_N}{\partial x'_1} & \cdots & \frac{\partial x_N}{\partial x'_N}
\end{array} \right) |.
\]

In the \(N\) dimensional case the corresponding formula is

\[
\prod_{n=1}^{N} \delta(x'_n - y'_n) = \prod_{n=1}^{N} \delta(x_n - y_n), J \left( \frac{\partial(x_1, \cdots, x_N)}{\partial(x'_1, \cdots, x'_N)} \right) |. 
\]

A standard example is the transformation from Cartesian to spherical coordinates in three dimensions where the variables are related by

\[
x = r \sin(\theta) \cos(\phi) \quad y = r \sin(\theta) \sin(\phi) \quad z = r \cos(\theta)
\]

The equation relating the volume elements is

\[
dx dy dz = r^2 \sin(\theta) d\theta d\phi.
\]

This gives

\[
\delta(x - x') \delta(y - y') \delta(z - z') = \frac{\delta(r - r') \delta(\theta - \theta') \delta(\phi - \phi')}{r^2 \sin(\theta)}.
\]

Note that when we multiply by the spherical volume element the terms in the denominator cancel and the delta functions leave all of the variables unchanged.