Lecture 35 - Topic 2: Boundary conditions - Laplace’s equation

In the previous lectures we discussed how to construct the singular part of the Green’s function for elliptic, hyperbolic and parabolic partial differential equations. In order to get unique solutions to physics problems it is necessary to add a solution to the homogeneous equation that fixes the boundary conditions.

This topic discusses the case of elliptic equations. For Laplace’s equation

$$\nabla^2 u(x) = 0$$

the problem is to find a solution of the partial differential equation that has a given value on a closed surface, or the normal derivative has a given value on a closed surface. The first set of boundary condition are called Dirichlet boundary conditions and the second set are called Neumann boundary conditions. It is also possible to look for solutions with mixed boundary conditions.

Recall that the divergence theorem has the form

$$\int_V dV \nabla \cdot A = \int_{\partial V} dS \hat{n} \cdot A$$

where $\partial V$ represents the boundary of the volume $V$. Using

$$A = u \nabla v \quad \text{and} \quad A = u \nabla v - v \nabla u$$

in the divergence theorem gives

$$\int_V dV \nabla \cdot (u \nabla v) = \int_{\partial V} dS \hat{n} \cdot (u \nabla v)$$

and

$$\int_V dV \nabla \cdot (u \nabla v - v \nabla u) = \int_{\partial V} dS \hat{n} \cdot (u \nabla v - v \nabla u).$$

The first equation can be expressed as

$$\int_V dV u \nabla^2 v = \int_{\partial V} dS \hat{n} \cdot (u \nabla v) - \int_V dV u \nabla \cdot \nabla v \quad (1)$$

while using this in the second gives

$$\int_V (u \nabla^2 v - v \nabla^2 u) = \int_{\partial V} dS \hat{n}(u \nabla v - v \nabla u). \quad (2)$$

We use these identities to prove some useful results about solutions of Laplace’s equation in three dimensions.

If $v$ is a solution of Laplace’s equation ($\nabla^2 v = 0$) and $u = 1$ then (1) gives

$$\int_{\partial V} dS \hat{n} \cdot \nabla v = 0. \quad (3)$$
A second useful identity is

\[
u(x) = \frac{1}{4\pi R^2} \int_{S_R} dS' u(x')\]

where \(S_R\) is a sphere of radius \(R\) centered at \(x\). To show this recall that the singular part of the Green’s functions for Laplace’s equation satisfies

\[\nabla^2 \frac{1}{r} = -4\pi \delta(x)\]

Use this in (2) with \(v = \frac{1}{r}\) and \(u\) a solution of Laplace’s equations to get

\[-4\pi u = \int_{\partial S_R} dS \hat{n}(u \nabla \frac{1}{r} - \frac{1}{R} \nabla u)\]

the second term on the right vanishes by (3) while the normal derivative of \(\frac{1}{r}\) on the surface of a sphere is \(-1/r^2\) which gives

\[u = \frac{1}{r \pi R^2} \int_{S_R} dS' u(x')\]

which proves the result. The third property is that if \(u\) is continuous on \(V\) and \(\partial V\) then it is either a constant or takes its maximum and minimum values on the boundary. This follows from the above result. Let \(x \in V\) but \(x \notin \partial V\) and assume that \(u\) takes on its maximum value at \(x\). Consider a sphere centered at \(x\) of sufficiently small radius so it is in \(V\) and all points on the surface \(u\) have a value less than \(u(x)\). Then

\[u(x) \leq \frac{1}{4\pi R^2} \int_{\partial S_R} dS' \max_{x' \in \partial S_R} u(x') = \max_{x' \in \partial S_R} u(x')\]

The only way that this can be satisfied if there are points on the boundary that are at least as large as \(u(x)\). This is true independent of \(R\) as long as the sphere is in the interior of \(V\). This means that \(u\) cannot have a maximum at \(x\). The same argument applies to minimum values with the inequalities reversed.

We can use these results to show that the Dirichlet boundary conditions result in a unique solution. If we have 2 solution of Laplace’s equation that agree on the boundary, their difference is a solution of Laplace’s equation that is identically 0 on the boundary. Since the solution takes its minimum and maximum values on the boundary, it must be zero everywhere.

For the Neumann boundary conditions assume there are 2 solutions of Laplace’s equation that have the same normal derivatives on the boundary. The difference, \(u\) is a solution of Laplace’s equation whose normal derivative vanishes on the boundary. Equation (1) with \(u = v\) gives

\[\int_V d\mathbf{x} |\nabla u|^2 = \int_{\partial V} dS u \frac{\partial u}{\partial n} = 0\]
which means that the two solutions can differ by at most a constant. These results are for the interior of closed boundaries. There are similar results for the volume external to the surface. They are derived by also accounting for the behavior of the functions at $\infty$. The strategy is to put the system in a large sphere, use the boundary conditions above between the original boundary and the interior of the large sphere and then let radius of the large sphere go to infinity. For Dirichlet boundary condition the condition at infinity is that the solution vanishes, for the Neumann boundary the condition is that $ru$ and $r^2|\nabla u|$ are bounded at $\infty$. 