Lecture 35 - Topic 3: Boundary conditions - parabolic equations

For this topic we consider the following equation

\[ \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{a^2} \frac{\partial u(x,t)}{\partial t} = 0. \]

We want solutions for \( t > 0 \) and \( x_1 < x < x_2 \) with boundary conditions

\[ u(x,0) = a(x) \quad u(x_1,t) = b(t) \quad u(x_2,t) = c(t). \]

We show that this set of boundary conditions gives a unique solution to the equation.

The strategy will be to show that the difference of two solutions satisfying these boundary conditions must vanish. To do this we first consider a rectangle with vertices \((x_1,0), (x_2,0), (x_1,T), (x_2,T)\). We don’t include the upper \( t = T \) as part of the boundary, since we do not put boundary condition on that side.

The boundary of the open rectangle is denoted by \( B \). Assume that there is a point \( x_0, t_0 \) in the interior of \( B \) where \( u(x,t) \) has a maximum. Let \( \Gamma \) denote the maximum value of \( u(x,t) \) on the boundary \( B \) of the rectangle. Then by assumption

\[ u(x_0,t_0) = \Gamma + \gamma \quad \gamma > 0. \]

Define a new function

\[ U(x,t) = u(x,t) + \eta(t_0 - t) \quad \eta > 0 \]

where we choose \( \eta \) so

\[ \eta|t - t_0| \leq \frac{\gamma}{2} \]

for all \( 0 < t < T \). Then —colored on \( B \) we have the inequality

\[ U(x,t) \leq \Gamma + \frac{\gamma}{2}. \]  \hspace{1cm} (1)

\( U(x,t) \) is continuous on the rectangle so it must have a maximum value (here the it is important that \( T \) is finite) at some point \((x_0', t_0')\) in the rectangle

\[ U(x_0', t_0') \geq U(x_0, t_0) = u(x_0, t_0) = \Gamma + \gamma > 0. \]  \hspace{1cm} (2)

In this case \( t_0' > 0 \) and \( x_1 < x_0' < x_2 \), since if \( t_0' \) in on the boundary (1) and (2) are incompatible.

The conditions for \( U(x_0', t_0') \) to have a maximum at \((x_0', t_0')\) are

\[ \frac{\partial U(x_0', t_0')}{\partial x} = 0 \]

\[ \frac{\partial U(x_0', t_0')}{\partial t} \geq 0 \]

\[ \frac{\partial^2 U(x_0', t_0')}{\partial x^2} \leq 0 \]

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The middle terms would normally be zero, unless the maximum occurs at \( t = T \) where the function should be increasing. Expressing these in terms of \( u(x, t) \)

\[
\frac{\partial U(x_0', t_0')}{\partial x} = \frac{\partial u(x_0', t_0')}{\partial x} = 0
\]

\[
\frac{\partial U(x_0', t_0')}{\partial t} = \frac{\partial u(x_0', t_0')}{\partial t} - \eta \geq 0
\]

\[
\frac{\partial^2 U(x_0', t_0')}{\partial x^2} = \frac{\partial^2 u(x_0', t_0')}{\partial x^2} \leq 0
\]

The differential equation gives

\[
\frac{\partial^2 u(x_0', t_0')}{\partial x^2} = \frac{1}{a^2} \frac{\partial u(x_0', t_0')}{\partial t} \geq \eta > 0
\]

which contradicts (3).

The means that the maximum value of \( u(x, t) \) must be on \( B \). A similar analysis means that the minimum value also occurs on the boundary. It there are two solutions satisfying the same boundary conditions their difference is 0 on the boundary, which is both the maximum and minimum value of the difference in the interior. This means that these boundary conditions give a unique solution. Since this holds for any \( T > 0 \) provided \( u(x, t) \) is finite at \( x = x_1 \), and \( x = x_2 \), the uniqueness holds for all \( t > 0 \).