Lecture 36 - Topic 1: Images

The method of images is a geometric way to solve differential equations with simple boundary conditions. The singular Green’s functions that were previously constructed satisfy

\[ L_x G_s(x, y) = 0 \quad \text{for} \quad x \neq y. \]

For linear differential equations linear combinations of solutions of the homogeneous equation are solutions of the homogeneous equations. It follows that

\[ L_x \sum_{n=1}^{N} G_s(x, y_n) = 0 \quad \text{for} \quad x \not\in \{y_k\}_{k=1}^{N}. \]

If we are looking for a solution of the homogeneous equation in a region \( R \) and we can find a collection of points \( \{y_k\}_{k=1}^{N} \) that are not in \( R \) and arrange them so

\[ u(x) := \sum_{n=1}^{N} G_s(x, y_n) \quad x \in R \]

satisfies the correct boundary conditions on \( R \), then by the uniqueness theorems, the \( u(x) \) above is the unique solution to the problem.

We can also solve for the Green’s function for \( x' \in R \)

\[ G(x, x') = G_s(x, x') + \sum_{n=1}^{N} G_s(x, y_n) \]

provided the \( \{y_k\}_{k=1}^{N} \) are not in \( R \) and can be arranged so \( G(x, x') \) satisfies the correct boundary conditions on the surface of \( R \). Typically this method only works for simple boundary conditions that have some symmetry.

Consider the example of Laplace’s equation in three dimensions. We have calculated the singular part of the Green’s function

\[ G_s(x, x') = -\frac{1}{4\pi} \frac{1}{|x - x'|}. \]

Consider the problem of finding the Green’s function for the interior of a sphere of radius \( R \) satisfying Dirichlet boundary conditions, \( u(r = R, \theta, \phi) = 0 \), which means that the Green’s function must vanish on the surface of the sphere.

To solve this we write

\[ G(x, x') = G = -\frac{1}{4\pi} \left( \frac{1}{|x - x'|} - \frac{k}{|x - x''|} \right) \]

If we can find a \( k \) and \( x'' \) with \( |x''| > R \) such that \( G(x, x') = 0 \) when \( |x| = R \), then the expression above is the desired Green function, since Dirichlet boundary conditions only allow for one solution.
By the symmetry of the problem we assume that
\[ x'' = \lambda x' \]
with so both vectors are on the same line through the origin. With this notation
\[
G(x, x') = -\frac{1}{4\pi} \left( \frac{1}{|x - x'|} - \frac{k}{|x - \lambda x'|} \right).
\]
This will vanish when |x| = R provided
\[
R^2 - 2\lambda R x' \cos(\theta) + \lambda^2 x'^2 = k^2 R^2 - 2k^2 R x' \cos(\theta) + k^2 x'^2.
\]
This requires
\[ k^2 = \lambda = \frac{x''}{x'} \]
and
\[
R^2 + (x'')^2 = k(R^2 + (x')^2) = x'' R^2 / x' + x' x''
\]
which is a quadratic equation for x''. It has 2 solutions, \( x'' = x' \) and \( x'' = R^2 / x' \). Only the second one is outside of the sphere. Thus
\[ x'' = R^2 / x' \quad k = R / x'. \]
This gives the following solution for the Green’s function that vanishes on the surface of a sphere of radius R:
\[
G(x, x') = -\frac{1}{4\pi} \left( \frac{1}{|x - x'|} - \frac{R}{x'} \frac{1}{|x - \frac{R^2}{x' x''}|} \right).
\]