Lecture 36 - Topic 3: Separation of variables - Laplace’s equation in spherical coordinates

In this section we apply the method of separation of variables to solve Laplace’s equation in spherical coordinates

\[ x = r \sin(\theta) \cos(\phi) \]

\[ y = r \sin(\theta) \sin(\phi) \]

\[ z = r \cos(\theta) \]

\[ \nabla = \hat{i} \left( \frac{\partial}{\partial x} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \frac{\partial}{\partial z} \right) + \hat{j} \left( \frac{\partial}{\partial x} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \frac{\partial}{\partial z} \right) + \hat{k} \left( \frac{\partial}{\partial x} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \frac{\partial}{\partial z} \right). \]

While tedious, it is straightforward to compute the partial derivatives. The algebra leads to a lot of cancellations. The next step is use the expression for the gradient in spherical polar coordinates to compute the Laplacian in spherical polar coordinates:

\[ \nabla^2 = \nabla \cdot \nabla = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial}{\partial r} \right] + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left[ \sin(\theta) \frac{\partial}{\partial \theta} \right] + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2}{\partial \phi^2}. \]

Since the purpose of this exercise is to show that Laplace’s equation is separable in spherical coordinates, the algebra of computing \( \nabla^2 \) is left as an exercise. As before we assume a solution of the form

\[ u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi). \]

It follows that

\[ 0 = \nabla^2 u(r, \theta, \phi) = \frac{\Theta(\theta)\Phi(\phi)}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial}{\partial r} R(r) \right] + \frac{R(r)\Phi(\phi)}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left[ \sin(\theta) \frac{\partial}{\partial \theta} \right] + \frac{R(r)\Theta(\theta)}{r^2 \sin^2(\theta)} \frac{\partial^2}{\partial \phi^2} \Phi(\phi). \]

Next multiply by \( \frac{r^2}{R(r)\Theta(\theta)\Phi(\phi)} \) to get

\[ 0 = \frac{1}{R(r)} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial R(r)}{\partial r} \right] + \frac{1}{\Theta(\theta)} \frac{\partial}{\partial \theta} \left[ \sin(\theta) \frac{\partial \Theta(\theta)}{\partial \theta} \right] + \frac{1}{\Phi(\phi)} \frac{\partial^2 \Phi(\phi)}{\partial \phi^2}. \]

The term on the left is a function of \( r \) while the sum of the other two terms do not depend on \( r \). It follows that

\[ c_1 = \frac{1}{R(r)} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial R(r)}{\partial r} \right]. \]
\[-c_1 = \frac{1}{\Theta(\theta) \sin(\theta)} \frac{\partial}{\partial \theta} \left[ \sin(\theta) \frac{\partial \Theta(\theta)}{\partial \theta} \right] + \frac{1}{\Phi(\phi) \sin^2(\theta)} \frac{\partial^2 \Phi(\phi)}{\partial \phi^2}\]

where \(c_1\) is a constant. The \(\theta\) and \(\phi\) equations can be separated by multiplying both sides of the second equation by \(\sin^2(\theta)\). This gives

\[-c_2 = \frac{1}{\Phi(\phi)} \frac{\partial^2 \Phi(\phi)}{\partial \phi^2}\]

\[c_2 = c_1 \sin^2(\theta) + \frac{\sin(\theta)}{\Theta(\theta)} \frac{\partial}{\partial \theta} [\sin(\theta) \frac{\partial \Theta(\theta)}{\partial \theta}].\]

The separation of variables has replaced the partial differential equation by three ordinary differential equations:

\[
\frac{d}{dr} \left[ r^2 \frac{dR(r)}{dr} \right] = c_1 R(r)
\]

\[
\sin(\theta) \frac{d}{d\theta} [\sin(\theta) \frac{d\Theta(\theta)}{d\theta}] + c_1 \sin^2(\theta) \Theta(\theta) = c_2 \Theta(\theta)
\]

\[
\frac{d^2 \Phi(\phi)}{d\phi^2} = -c_2 \Phi(\phi).
\]

Next we consider boundary conditions. For a single-valued analytic solution \(\Phi(\phi)\) must be periodic, so

\[\Phi(\phi) = e^{im\phi},\]

which implies \(c_2 = m^2\) where \(m\) is an integer.

The radial equation is

\[r^2 R'' + 2r R' - c_1 R = 0.\]

The solution of this by inspection has the form \(r^\lambda\) where

\[\lambda(\lambda - 1) + 2\lambda - c_1 = 0\]

\[\lambda^2 + \lambda - c_1 = 0\]

\[\lambda_{\pm} = \frac{1}{2} \left(-1 \pm \sqrt{1 + 4c_1}\right).\]

We will see that the allowed values of \(c_1\) are determined by the \(\theta\) equation, so we consider that equation first. For the \(\theta\) equation let \(u = \cos(\theta)\) which gives

\[du = -\sin(\theta)d\theta\]

\[
\sin(\theta) \frac{d}{d\theta} = -\sin^2(\theta) \frac{d}{du} = (u^2 - 1) \frac{d}{du}\]

which leads to

\[
\frac{d}{du} (u^2 - 1) \frac{d\Theta}{du} + \frac{m^2}{1-u^2} \Theta = c_1 \Theta
\]

or

\[
(1 - u^2) \Theta'' - 2u \Theta' + \left(c_1 - \frac{m^2}{1-u^2}\right) \Theta = 0
\]

This shows that Laplace’s equation is also separable in spherical coordinates.