Lecture 37 - Topic 1: Completeness of the spherical harmonics.

We start by expressing a continuous function of the form $f(\theta, \phi)$ in Cartesian coordinates

$$f(\theta, \phi) = g(x, y, z)$$

Functions of the angles only are functions on the unit sphere so $f(\theta, \phi)$ depends on $x, y, z$ constrained so $|x| = 1$. By the Weierstrass theorem $g(x, y, z)$ can be uniformly approximated by polynomials of the form

$$g_n(x, y, z) = \sum_{m_1 \leq n} g_{m_1 m_2 m_3} x^{m_1} y^{m_2} z^{m_3}$$

where in this case $|x| = 1$. We call the functions of the form, $x^{m_1} y^{m_2} z^{m_3}$, where $\sum k m_k = n$ monomials of order $n$. The Weierstrass theorem implies that the independent monomials are complete. Next we want to determine how many of these monomials are independent of each other and of lower order monomials when we impose the constraint. To understand this note

$$x^2 + y^2 + z^2 = 1$$

means that $z^2$ can be expressed in terms of 1 (a lower order monomial) and $x^2$ and $y^2$. In general the constraint means that whenever we get factor of $z^2$ it can be eliminated by the constraint. The means the independent monomials of order $n$ can be taken as

$$x^{m_1} y^{m_2} m_1 + m_2 = n \quad x^{m_1} y^{m_2} z \quad m_1 + m_2 + 1 = n$$

There are $n + 1$ possibilities of the first type and $n$ for the second type, giving $2n + 1$ linearly independent monomials of order $n$.

The completeness of the spherical harmonics follows because the $Y_{l}^m$ are $2l + 1$ independent linear combinations of monomials of order $l$. The independence follows from the orthogonality. To see that they are monomials recall

$$P_{l}^m(u) = \frac{(-1)^m l!}{2^l (1 - u^2)^{l/2}} \frac{d^{l+m}}{du^{l+m}} (1 - u^2)^{l}$$

The factor $(1 - u^2)^{m/2} = \sin^m(\theta)$ while the term with the derivatives is a linear combination terms of the form $u^k(1 - u^2)^n$ with $k + n = l - m$, which has the form $\cos^k(\theta) \sin^{2n}(\theta)$ which behaves like $z^k(x^2 + y^2)^n$. The factor $\sin^m(\theta)e^{im\phi} = (x + iy)^m$ which is also a homogeneous polynomial.

The demonstrates that the spherical harmonics $Y_{l}^m$ are $2l + 1$ independent monomials of order $l$, which means the collection of all spherical harmonics are a complete set of function of $\theta$ and $\phi$. 

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