Lecture 37 - Topic 2: Angular equation revisited.

When we first discussed separation of variables we had a general solution where the separation constant $c_1$ was not fixed. Then we considered the special case where $c_1 = l(l + 1)$ with $l$ a non-negative integer, which led to the spherical harmonics. Next we want to return to understand the conditions that fix $l$.

The $\theta$-differential equation can be expressed in the form

$$(1 - u^2)\Theta'' - 2u\Theta' - \frac{m^2}{1 - u^2}\Theta = -c_1\Theta$$

This can be written as

$$\frac{d}{du} \left( (1 - u^2) \frac{d\Theta}{du} \right) - \frac{m^2}{1 - u^2}\Theta = -c_1\Theta$$

from which it can be seen that

$$L_u = \frac{d}{du} \left( (1 - u^2) \frac{d}{du} \right) - \frac{m^2}{1 - u^2}$$

it is a self-adjoint operator with weight 1. It follows that

$$\int_{-1}^{1} (P_{m}^{\lambda \ast}(u)L_u P_{m}^{\lambda'} - P_{m}^{\lambda'}(u)(L_u P_{m}^{\lambda \ast})) du = (1 - u^2)(P_{m}^{\lambda \ast}(u) \frac{d}{du} P_{m}^{\lambda'} - P_{m}^{\lambda'}(u) \frac{d}{du} P_{m}^{\lambda \ast})|_{u = -1}^{u = 1}$$

this vanishes as long as $P_{m}^{\lambda}(u)$ and $P_{m}^{\lambda'}(u)$ are finite at the endpoints. (Here we change notation slightly replacing $l$ by $l(l + 1) = \lambda$).

It follows that

$$(\lambda - \lambda') \int_{-1}^{1} P_{m}^{\lambda \ast}(u)P_{m}^{\lambda'}(u) du = 0$$

If $P_{m}^{\lambda}(u)$ were continuous and $\lambda \neq l(l + 1)$ then by the completeness of the spherical harmonics

$$P_{m}^{\lambda}(u)e^{im\phi} = \sum_{lm'} a_{lm'} Y_{lm'}$$

The coefficients are

$$a_{lm'} = \int \sin(\theta) d\theta d\phi P_{m}^{\lambda}(\cos(\theta))e^{im\phi}Y_{lm'}^{\ast}(\theta, \phi) = 0$$

due to the orthogonality of $P_{m}^{\lambda}(u)$ and $P_{m}^{l(l + 1)}(u)$, unless $\lambda = l(l + 1)$ for some $l$ which means $P_{m}^{\lambda}(u) = 0$. The only other possibility is that $P_{m}^{\lambda}(u)$ is the second solution to the differential equation with $\lambda = l(l + 1)$. If this were continuous it could be expanded as above which would contradict that fact that it is an independent solution of the equation.

This allows us to conclude that the finite solutions have $\lambda = l(l + 1)$ with $l$ a non-negative integer.