Lecture 1

Group Theory | Group representations

* Group theory is the mathematics of symmetries. Consider the eigenvector problem

\[ H |v\rangle = \lambda |v\rangle \]

Assume that \( U \) is an operator that commutes with \( H \)

\[ [U,H] = 0 \]

Then

\[ H U |v\rangle = U H |v\rangle = U (\lambda |v\rangle) = \lambda U |v\rangle \]

This means that \( U |v\rangle \) is also an eigenvector of \( H \) with the same eigenvalue.

Example: \( H = \) Hamiltonian, \( U = P \)

= space reflection. \( [P,H] = 0 \)

Means that the space reflection of an eigenvector of \( H \) is also an eigenvector of \( H \) with the same eigenvalue.
Definition: A group consists of a set of group elements \( G \) and a rule that assigns a new group element to an ordered pair of group elements

\[ g_3 = g_1 \cdot g_2 \]

with the following properties:

1. There is an identity element \( e \in G \) with the property
   \[ eg = ge = g \quad \forall g \in G \]

2. Existence of inverse
   \[ \forall g \in G \exists g' \text{ satisfying} \]
   \[ gg' = g'g = e \]

3. Associativity
   \[ \forall g_1, g_2, g_3 \in G \]
   \[ g_1(g_2g_3) = (g_1g_2)g_3 \]
examples

1. \( G = \{ e \} \)
   
   a set consisting of the identity is a group. Clearly, \( e \) is its own inverse.

2. \( G = \{ e, a, a^{-1} \} \)

   where
   
   \[
   e^2 = e \
   a = ea = ae \
   a^{-1} = ea^{-1} = a^{-1}a^{-1} \rightleftharpoons \text{identity element} \rightleftharpoons e \
   a^{-1} = a^{-1}a = e \
   a^2 = a^{-1} (a^{-1})^2 = a
   \]

   For finite groups this may product can be put in the form of a table

   \[
   \begin{array}{c|ccc}
   \hline
   & e & a & a^{-1} \\
   \hline
   e & e & a & a^{-1} \\
   a & a^{-1} & e & e \\
   a^{-1} & e & e & a \\
   \hline
   \end{array}
   \]

   This group is called the cyclic group of order 3, \( \mathbb{Z}_3 \)

   \[
   \begin{align*}
   e: & (1, 2, 3) \rightarrow (1, 2, 3) \\
   a: & (1, 2, 3) \rightarrow (3, 1, 2) \\
   a^{-1}: & (1, 2, 3) \rightarrow (2, 3, 1)
   \end{align*}
   \]
<table>
<thead>
<tr>
<th>3</th>
<th>$\mathbb{Z}_4$</th>
<th>$e$</th>
<th>$a_1$</th>
<th>$a_2$</th>
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<tbody>
<tr>
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<td>$a_1$</td>
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<td>$a_3$</td>
</tr>
</tbody>
</table>

Note that every row and every column has different elements. This is needed so every element to have an inverse.

In this case, the multiplication table is symmetric.

$$ab = ba.$$ This is not a general property of groups.

Groups satisfying

$$ab = ba \quad \forall a, b$$

are called abelian groups.

Groups that do not have this property are called non-abelian groups.
Example of a non-abelian group 2x2 real matrices with det = 1

* Since det = 1 every element has an inverse

\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}
\]

\[
\begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{pmatrix}
\]

This group is called \( SL(2\mathbb{R}) \)

\[
S = 0 \quad \text{det} \quad A = 1
\]

\[L = 0 \quad \text{matrix group}\]

\[2 = \text{dimension of matrix \((2 \times 2)\)}\]

\[R = \text{real}\]

* \( \mathbb{Z}_3 \mathbb{Z}_4 \) have a finite number of elements. The number of elements of \( G \) is called the order of the group.

\[O(\mathbb{Z}_3) = 3 \quad O(\mathbb{Z}_4) = 4\]

The order of \( SL(2\mathbb{R}) \) is infinite.
A group does not have to be infinite to be non-abelian. A simple example is the permutation group on 3 objects.

\[
\begin{align*}
C & : (1 \ 2 \ 3) \\
& \quad (1 \ 2 \ 3)
\end{align*}
\]
\[
\begin{align*}
P_{12} & = (1 \ \ 2 \ \ 3) \\
& \quad (2 \ \ 1 \ \ 3)
\end{align*}
\]
\[
\begin{align*}
P_{23} & = (1 \ \ 2 \ \ 3) \\
& \quad (1 \ \ 3 \ \ 2)
\end{align*}
\]
\[
\begin{align*}
P_{31} & = (1 \ \ 2 \ \ 3) \\
& \quad (3 \ \ 2 \ \ 1)
\end{align*}
\]
\[
\begin{align*}
\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} & = P_{23} P_{12} \\
\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} & = P_{12} P_{23}
\end{align*}
\]

We see that \( P_{23} P_{12} \neq P_{12} P_{23} \) so this is a non-abelian group of order 6. The book uses the notation:

\[
\begin{align*}
a_1 & = P_{23} P_{12} & a_2 & = P_{12} P_{23} \\
a_3 & = P_{12} & a_4 & = P_{23} & a_5 & = P_{31}
\end{align*}
\]
for the multiplication table

\[ P_{12} P_{12} = e \quad (P_{12} P_{23})^{-1} = P_{23} P_{12} \]
\[ P_{23} P_{23} = e \quad (P_{23} P_{12})^{-1} = P_{12} P_{23} \]
\[ P_{31} P_{31} = e \]

\[ P_{12} P_{23} = P_{23} P_{31} = P_{31} P_{12} = a_2 \]
\[ P_{23} P_{12} = P_{31} P_{23} = P_{12} P_{31} = a_1 \]

Homework - work out the multiplication table in \( S_3 \) - show

(1) every row and column has different elements

(2) the matrix is not symmetric about the diagonal

Group representations

What is important in physics problems are group representations

If \( G \) is a group a group representation is a mapping \( D: G \to \text{space of linear operators} \) satisfying

(1) \( D(e) = I \)

(2) \( D(g_1 g_2) = D(g_1) D(g_2) \)
example: representations of \( \mathbb{Z}_3 \)

\[
D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
D(a) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}
\]

\[
D(a^2) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}
\]

Check that \( D(a_1) D(a_2) = D(a_1 a_2) \) for homework. This is a 2-dimensional representation.

Another representation of \( \mathbb{Z}_3 \) is

\[
D(e) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad D(a) = e^{\frac{2\pi i}{3}} \quad D(a^2) = e^{\frac{4\pi i}{3}}
\]

This is a one-dimensional representation.

There is a special representation called the regular representation.

In this representation, the group elements are treated as orthonormal vectors.

\( |e\rangle, |a\rangle, |a^2\rangle \)
we define
\[ D(g)|q^i> = 1gq^i > \]
\[ D(e)|e> = |e> \]
\[ D(a)|e> = |a> \]
\[ D(a^i)|e> = |a^i> \]
\[ D(e)|a^i> = |a_i> \]
\[ D(a)|a^i> = |a^i> \]
\[ D(a^i)|a^i> = |e> \]

This gives
\[
D(e) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
\[
|e> = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]
\[
|a> = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]
\[
|a^i> = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

\[
D(a) = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]
\[
D(a^i) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]
Equivalent representations

If \( G \) is a group and \( D(g) \) is a representation of \( G \) and \( S \) is an invertible matrix

\[
D'(g) = SD(g)S^{-1}
\]

is an equivalent representation of \( G \)

Note

\[
D'(e) = SD(e)S^{-1} = SS^{-1} = I
\]

\[
D'(g_1)D'(g_2) = SD(g_1)S^{-1}SD(g_2)S^{-1}
= S(DD(g_1))S^{-1} = SD(g_1g_2)S^{-1}
\]

These 2 representations differ by a change of basis -

example - in the regular representation

\[
|e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |a\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |a^{-1}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
D'(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D'(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D'(a^{-1}) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

here the matrices \( D'(a) = D(a^{-1}) \), \( D'(a^{-1}) = D(a) \)
These are equivalent representations of $\mathbb{Z}_3$.

2 representations of the same group that are not equivalent are called inequivalent representations.

A representation $D(g)$ on a linear space $L$ is called an irreducible representation if it has no non-trivial invariant subspaces.

**Meaning:** Let $P$ project on a proper subspace of $L$

$$D(g)P = PD(g)P \quad \forall g \in G$$

Trivial means $P \neq I = 0$.

A representation is completely reducible if it is equivalent to a representation that is block diagonal — with each block being an irreducible representation.
This means

\[ SD_1(q) S^{-1} = \begin{pmatrix} D_1(q) & 0 & 0 & \cdots \\ 0 & D_2(q) & 0 & \cdots \\ 0 & 0 & D_3(q) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad A_{960} \]

where each \( D_i(q) \) is irreducible.

The block diagonal representation is called the direct sum representation

\[ D_1(q) = D_1(q) \oplus D_2(q) \oplus D_3(q) \]

which indicates that each block acts on an orthogonal subspace.

Example - consider the regular representation of \( \mathbb{Z}_3 \)

\[ \omega \equiv e^{2\pi i / 3} \quad \omega^3 - 1 = 0 = (\omega - 1)(1 + \omega + \omega^2) \]

\[ \omega^{-1} = \omega^2 \quad \omega^3 = 1 \quad \text{if } \omega \neq 1 \]

\[ S = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad S^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \]
direct calculation gives

$$S' D(e) S = D'(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S' D(a) S = D'(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$$S' D(a') S = D'(a') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

we see that this becomes a direct sum of 3 irreducible representations.

In this case -1 is trivial, the other two have elements 1, \(\omega\), \(\omega^2\)

$$D(e) = D(a) = D(a') = 1$$

$$D(e) = 1 \quad D(e) = \omega \quad D(a') = \omega^2$$

$$D(e) = 1 \quad D(a) = \omega^2 \quad D(a') = \omega$$

Role of transformation groups in quantum theories

$$|v\rangle \rightarrow |v'\rangle = D(q) |v\rangle$$

$$D(q):$$ quantum Hilbert space - to itself

consider the case where \(D(q)\) is unitary - (we will show that any representation of a finite group is equivalent to
a unitary representation

\[ \langle v' | w' \rangle = \langle v | D^+ (g) D(g) | w \rangle = \langle v | w \rangle \]

unitary representations preserve inner products

If \[ \left[ D(g), H \right] = 0 \] then

\[ H | v \rangle = \lambda | v \rangle \]

\[ H D(g) | v \rangle = D(g) H | v \rangle = \lambda D(g) | v \rangle \]

This means that for every \( g \in \mathbb{G} \),

\( D(g) | v \rangle \)

is an eigenvector of \( H \) with eigenvalue \( \lambda \).

This means that we can choose eigenvectors that transform like representations of a group

\[ | g \rangle \equiv D(g) | v \rangle \]

example: space reflection

\( p = \text{open cut} \)

\( p^2 = e \)

This group has 2 elements

\[
\begin{array}{c|ccc}
  & e & p & e \\
\hline
  e & e & p & e \\
  p & p & e & p \\
\end{array}
\]

with the multiplication table
in the regular representation

\[ D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]
\[ D(p) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

These can both be diagonalized by

\[ D'(q) = S D(q) S' \]

\[ S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad S' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]

\[ S D(e) S' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ S D(p) S' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \]

\[ = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

There are 2 representations - the identity representation

\[ D'(e) = 1 \quad D'(I) = 1 \]

and

\[ D'(p) = -1 \quad D'(e) = 1 \]

These are realized by wave functions that are symmetric or antisymmetric with respect to space reflection.
example: irreducible representation of the permutation group on 3 objects:

\[
\begin{align*}
D(e) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & D(a_1) &= D(\sigma_1 \sigma_2) = \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \\
D(a_2) &= D(\sigma_1 \sigma_3) = \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} & D(\sigma_2) &= \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \\
D(\sigma_3) &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & D(\sigma_4) &= \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \\
D(\sigma_5) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{align*}
\]

since the permutation group on 3 objects is non-abelian - some of the irreducible representations must be matrices (numbers commute).

In general - a general representation can only be block diagonalized by a similarity transformation - it can't be completely diagonalized.

Comments on representations:

* Not every reducible representation is completely reducible.

Example: Integers under addition

\[
\begin{align*}
D(n) &= \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \\
D(n)D(m) &= \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m+n \\ 0 & 1 \end{pmatrix}
\end{align*}
\]
Let \( P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) \( P^2 = P \) \( \omega = 1 - P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \)

\( D(n)P = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = P \)

\( D(n) \) maps the range of \( P \) to the range of \( \omega \).

\( PD(n)P = D(n)P = P. \)

On the other hand,

\( PD(n)\omega = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \)

\( = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq 0 \)

This means that \( D(n) \) connects the \( P \) and \( \omega \) block - so it is not block diagonal and completely reducible.

* every representation of a finite group is equivalent to a unitary representation

Proof: Let \( D(g) \) be a representation of \( G \)

Define \( S = \sum_{g \in G} D(g)^\dagger D(g) \)

where the sum is over all group elements. By construction \( S = S^+ \)

we also note

\( \langle u | V | S | V | u \rangle = \sum_{g \in G} \langle u | V | D(g)^\dagger D(g) | V | u \rangle = \)
\[ = \langle \nu | D(e) D(e) | \nu \rangle + \sum_{g \neq e} \| D(g) | \nu \rangle \|^2 \]
\[ = \| \nu \|^2 + \sum_{g \neq e} \| D(g) | \nu \rangle \|^2 > 0 \]

It follows that \( S \) is a positive hermitian operator, since the eigenvalues are all \( \geq 1 \) by the above calculation, \( S \) has a positive square root, \( S^{1/2} \), which also has an inverse.

Define
\[ D'(g) = S^{1/2} D(g) S^{-1/2} \]
\[ D'^\dagger (g) = S^{-1/2} D^\dagger (g) S^{1/2} \quad (\text{since } S^{1/2} = S^{1/2} \dagger) \]

(recall positive square root of positive operator are positive and hermitian)

\[ D(g') D(g) = S^{-1/2} D(g) S \quad S D(g') S^{-1/2} = \]
\[ = S^{-1/2} D(g) S D(g') S^{-1/2} = \]
\[ = \sum_{g' \neq g} D(g') D(g) D(g') D(g) S^{-1/2} = \]
\[ = \sum_{g' \neq g} D(g') D(g') D(g') D(g') S^{-1/2} = \]
\[ = S^{-1/2} \sum_{g' \neq g} D(g') D(g') D(g') D(g') S^{-1/2} \quad \text{let } g'' = g' g \]
\[ \geq g'' = 2 g' \]
\[ = S^{-1/2} S S^{-1/2} = I \]

This proves the result.
x every reducible representation of a finite group is completely reducible.

by the previous result we can restrict considerations to unitary representations.

If \( D(g) \) is reducible then there is a projection operator \( P \) satisfying

\[
P D(g) P = D(g) P \quad \forall g \in G
\]

Taking adjoint

\[
P D^*(g) P = P D^*(g)
\]

by unitarity

\[
D^*(g) = D^*(g') = D(g')
\]

\[
P D(g') P = P D(g')
\]

since \( g \) is arbitrary

\[
P D(g) P = P D(g)
\]

\[
D(g)(1-P) = D(g) - D(g) P
\]

\[
D(g) - P D(g) P = D(g) - P D(g)
\]

\[
(1-P) D(g)
\]

multiply by \( 1-P \) on the right

\[
D(g)(1-P) = (1-P) D(g)(1-P)
\]
This means that if \( D(g) \) is reducible, it can be block diagonalized - if the blocks are reducible this can be repeated until all blocks are irreducible - proving the result.