Theorem: Classical orthogonal polynomials are solutions of a second order differential equation.

Consider

\[ \frac{1}{w} \frac{d}{dx} \left( ws \frac{dC_n}{dx} \right) = s \frac{d^2C_n}{dx^2} + K_{c}\frac{dC_n}{dx} = \text{degree } n \]

It follows that this can be expanded as a series of polynomials with degree \( \leq n \)

\[ \frac{1}{w} \frac{d}{dx} (ws \frac{dC_n}{dx}) = \sum_{n=0}^{n} A_n C_n(x) \]

Multiply by \( C_k(x)\omega(x) \)

\[ \int_{a}^{b} C_k(x) \frac{d}{dx} (ws \frac{dC_n}{dx}) = 0_k h_k \]

Integrate by parts twice

\[ -\int_{a}^{b} \frac{dC_k}{dx} (ws \frac{dC_n}{dx}) + C_k (ws \frac{dC_n}{dx}) \bigg|_{a}^{b} = \]

\[ \int \frac{d}{dx} (ws \frac{dC_k}{dx}) C_n + \frac{dC_k}{dx} (ws C_n) \bigg|_{a}^{b} = \]

\[ \int \omega(x) \left\{ \frac{d}{dx} (ws \frac{dC_k}{dx}) \right\} C_n \]

\[ K_{c} \frac{dC_k}{dx} + s \frac{d^2C_k}{dx^2} = \text{degree } k \]

Vanishes by orthogonality if \( k \neq n \)
The surviving term corresponds to \( k = n \)

\[
\frac{1}{\omega} \frac{d}{dx} \left( \omega s \frac{dc_n}{dx} \right) = a_n C_n
\]

or

\[
K_1 \frac{dc_n}{dx} + \gamma \frac{d^2c_n}{dx^2} = a_n C_n
\]

This is the desired second order equation. To find an arbitrary multiply by \( \omega \) \( C_n \) and integrate

\[
\int K_1 \frac{dc_n}{dx} \, C_n \, \omega + \int \gamma \frac{d^2c_n}{dx^2} \, C_n \, \omega = a_n \, h_n
\]

\[
\int K_1 \frac{dc_n}{dx} \, n \cdot C_n \cdot C_n \, \omega + \int \gamma \frac{d^2c_n}{dx^2} \, C_n \, \omega = a_n \, h_n
\]

\[
\int \frac{1}{2} s'' n(n-1) C_n^2 \, \omega + \int \gamma \frac{d^2c_n}{dx^2} \, C_n \, \omega = a_n \, h_n
\]

This gives

\[
K_1 \frac{dc_n}{dx} \, n \, h_n + \frac{1}{2} s'' n(n-1) h_n = a_n h_n
\]

Canceling the \( h_n \)

\[
a_n = K_1 \frac{dc_n}{dx} \, n + \frac{1}{2} (n)(n-1) s''
\]
Given a basis of orthogonal polynomials $|n\rangle$. A general expansion

$$|q\rangle = \sum_{n=0}^{\infty} q_n |n\rangle$$

If we consider

$$\|1,f\rangle - |q\rangle\|^2 = \langle f | f\rangle - \langle f | 1,f\rangle - \langle 1,f | f\rangle + \langle 1,f | 1,f\rangle$$

$$= \langle f | f\rangle - 2 \langle f | 1,f\rangle q_n - q_n^* \langle 1,f | f\rangle + 2 |q_n|^2$$

and subtract $\sum |k\rangle \langle k|$, we get

$$= \langle f | f\rangle - \sum_{n=0}^{\infty} \langle f | k\rangle \langle k | f\rangle - q_n \langle 1,f | f\rangle + \sum_{n=0}^{\infty} |q_n|^2$$

$$= \langle f | f\rangle - \sum_{n=0}^{\infty} \langle f | k\rangle \langle k | f\rangle - q_n \langle 1,f | f\rangle + \sum_{n=0}^{\infty} |q_n|^2$$

positive by Bessel's inequality.

In order to minimize the $L^2_w$ norm $q_n = \langle n | f\rangle$ (i.e., the Fourier coefficients).

Since polynomials are complete

$$\|1,f\rangle = \sum q_n^* |n\rangle \rightarrow |f\rangle$$

can be found that are cauchy converge to a vector

$$\|1,f\rangle - |f\rangle\|^2 = \|1,f\rangle - |f\rangle\|^2 + \sum |q_n|^2 - |q_n|^2$$
\[ \|f_n\|_2^2 - \|f\|_2^2 - 2|\langle f_n, f \rangle| = \|f_n\|_2 - \|f\|_2 > 0 \]

If \( |\langle f_n, f \rangle| \to |\langle f, f \rangle| \) then \( |\langle f_n, f \rangle| \to |\langle f, f \rangle| \)

On \( |\langle f_n, f \rangle| \) is also Cauchy.

The only issue is that the orthogonal polynomials converge in the mean to an abstract vector, the Fourier sum converges to a vector that could differ from the vector being expanded on a set of measure 0; so the convergence of

\[ \lim_{n \to \infty} |\langle x, f_n \rangle < h, f \rangle| = f'(x) \]

which could differ from \( f'(x) \) on a set of measure 0.

*Note: Weierstrass converges pointwise in continuous functions but it does not necessarily agree with the basis function expansion.*
Fourier series

Let \( f(\theta) \) be continuous and periodic on \( [-\pi, \pi] \).

Define

\[
g(\theta) = f(\theta)
\]

Let

\[
x = r \cos \theta, \quad y = r \sin \theta
\]

\[
h(x, y) = g(r(x, y) \theta(\theta))
\]

because \( \theta \) is periodic \( h(x, y) \) is a continuous single valued continuous function on \( -1 \leq xy \leq 1 \).

Remark: the proof given in the book of the Weierstrass theorem generalizes to vector variables.

\[
\forall \epsilon > 0 \exists N_0 \text{ such that for } N > N_0 \text{ there are coefficients } h_{nm} \text{ such that}
\]

\[
| h(x, y) - \sum_{m=0}^{N} \sum_{n=0}^{N} h_{mn} x^m y^n | < \epsilon, \quad -1 \leq xy \leq 1
\]

we can express this in polar coordinates.
Note that
\[ \int_{-\pi}^{\pi} (e^{i\omega \theta})^* (e^{i\omega \theta}) d\theta = \begin{cases} 2\pi & \text{if } m=n \\ \frac{1}{i(n-m)} e^{i(n-m)\theta} \bigg|_{-\pi}^{\pi} = 0 & \text{if } m \neq n \end{cases} \]

If we define
\[ \langle \Omega | n \rangle = \frac{1}{\sqrt{2\pi}} e^{i\omega n} \quad n = -\infty \ldots \infty \]

Then we have
\[ \int_{-\pi}^{\pi} \langle m | \omega \rangle \langle \omega | n \rangle d\omega = \delta_{mn} \]

\[ \langle \Omega | n \rangle \] is an orthonormal basis for periodic functions.

\[ |f(\omega) - \sum_{n=-\infty}^{\infty} g_n e^{in\omega}| < \epsilon \Rightarrow \]
\[ \int_{-\pi}^{\pi} |f - \sum_{n=-\infty}^{\infty} g_n e^{in\omega}|^2 d\omega < \epsilon^2 \times 2\pi \]

These infinite sums converge to
\[ \sum_{n=-\infty}^{\infty} \left\langle f | \Omega_n \right\rangle \langle \Omega_n | f \rangle = 0 \]

In general,
\[ \langle \tilde{f} | = \sum_{n=-\infty}^{\infty} \left\langle \Omega_n | f \right\rangle \left\langle f | \Omega_n \right\rangle \]

\[ \| \tilde{f} \| = \| f \| = 0 \]
we can also construct real basis functions:

\[ <\theta_{1n^+}> = \frac{1}{\sqrt{2}} ( <\theta_{1n}> + <\theta_{1-n}> ) = \]

\[ \frac{1}{\sqrt{8}} \cos(n\omega) \]

\[ <\theta_{1n^-}> = -\frac{i}{\sqrt{2}} ( <\theta_{1n}> - <\theta_{1-n}> ) = \]

\[ \frac{1}{\sqrt{8}} \sin(n\omega) \]

\[ <\theta_{10}> = \frac{1}{\sqrt{2\pi}} \]

In general we write:

\[ <\theta_{1f}> = \sum_{n=-\infty}^{\infty} <\theta_{1n}> <\theta_{1f}> + \]

\[ <\theta_{10}> <\theta_{1f}> + \sum_{n=1}^{\infty} <\theta_{1n^+}> <\theta_{1n+1f}> + \]

\[ \sum_{n=1}^{\infty} <\theta_{1n^-}> <\theta_{1n-1f}> \]
we have shown that $\langle \sin \rangle$ is an orthonormal basis in the space of continuous functions on $[-\pi, \pi]$

It is easy to show that it is also a basis for non-periodic functions

let $f_\pm = f(\pm \pi)$

define

$$f_N(\theta) = \begin{cases} f(\theta) & -\pi \leq \theta \leq \pi - \frac{\pi}{N} \\ \begin{array}{cc} f(\pi - \frac{\pi}{N}) + (\theta - \pi + \frac{\pi}{N}) \frac{\pi}{N} (f_+ - f(\pi - \frac{\pi}{N})) & \end{array} & \end{cases}$$

$$|f(\theta) - f_N(\theta)| = \text{norm 0 and bounded on a set of width } \frac{\pi}{N}$$

$$\|f - f_N\| < \max|\Delta s| \cdot \frac{\pi}{N} \to 0$$

$\langle \sin \rangle$ is periodic and can be approximated by a polynomial

$$\|f - P\| \leq \|f - f_N\| + \|f_N - P\| < \frac{\pi}{2} + \frac{\pi}{N}$$
In this case the convergence is only in the mean.

Note \[ \{ \sin^+ \mid 0 \} \] even \( \in [-\pi, \pi] \)
\[ \{ \sin^- \} \] odd

Let \( f(\theta) \) be continuous and defined on \( [0, \pi] \) - define

\[
\begin{align*}
\frac{f_+}{f_-}(\theta) &= \begin{cases} 
  f(\theta) & 0 \leq \theta \leq \pi \\
  f(\theta) & -\pi < \theta < 0 
\end{cases} \\
\frac{f_0}{-f_0}(\theta) &= \begin{cases} 
  f(\theta) & 0 \leq \theta \leq \pi \\
  f(\theta) & -\pi < \theta < 0 
\end{cases}
\end{align*}
\]

Note \( \left< n-1 f_m \right> = \left< n+1 f_m \right> = \left< 0 f_m \right> = 0 \)

\[
\begin{align*}
\frac{f_+}{f_-}(\theta) &= \sum_{n=1}^{\infty} \left< 0 \sin^+ \right> \left< n^+ \sin^+ \right> + \left< 0 \sin^- \right> \left< n^+ \sin^- \right> \\
\frac{f_0}{-f_0}(\theta) &= \sum_{n=1}^{\infty} \left< 0 \sin^- \right> \left< n^- \sin^- \right>
\end{align*}
\]

These functions agree \((0, \pi)\) on \([0, \pi]\), thus the functions \((0, \pi)\) we can expand \( \cos n \theta \) or \( \sin n \theta \).
while in general the Fourier series converges in the mean, the Weierstrass theorem suggests that there are conditions when the convergence is uniform, while there are many special cases.

Consider the finite expansion

$$
\sum_{n=-N}^{N} \langle \phi | \psi \rangle = \\
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \sum_{n=-N}^{N} e^{in(\theta-\theta')} \right] f(\theta') d\theta' = \\
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{n=1}^{N} e^{in(\theta-\theta')} + \sum_{n=-N}^{-1} e^{-in(\theta-\theta')} \right) f(\theta') = \\
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{n=0}^{N} e^{i(n\theta)(\theta-\theta')} + \sum_{n=-N}^{-1} e^{-i(n\theta)(\theta-\theta')} \right) f(\theta') = \\
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{1-e^{i(\theta-\theta')}}{1-e^{-i(\theta-\theta')}} + \frac{1-e^{-i(\theta-\theta')}}{1-e^{i(\theta-\theta')}} \right) f(\theta') = \\
\Delta \theta = \Theta - \theta'

\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ e^{i\frac{\theta}{2}\Delta \theta} \frac{\sin\left(\frac{\theta}{2}\Delta \theta\right)}{\sin(\Delta \theta/2)} + e^{-i\frac{\theta}{2}\Delta \theta} \frac{\sin\left(\frac{\theta}{2}\Delta \theta\right)}{\sin(\Delta \theta/2)} - 1 \right] f(\theta') d\theta'

Next use

$$
\sin \alpha \cos \beta = \frac{1}{2} \left( \sin(\alpha-\beta) + \sin(\alpha+\beta) \right)
$$
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{\sin \left( \frac{n\phi}{2} \right)}{\sin \left( \frac{\phi}{2} \right)} + \frac{\sin \left( (n+\frac{1}{2})\phi \right)}{\sin \left( \frac{\phi}{2} \right)} - 1 \right) f(\phi') = \\
\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin \left( (n+\frac{1}{2})(\theta - \phi') \right)}{\sin \left( \theta' - \phi \right)} f(\phi')
\]

Let \( \phi = \theta' - \phi \)

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin \left( (n+\frac{1}{2})\phi \right)}{\sin \left( \phi' \right)} f(\phi + \phi') d\phi
\]

Because this is periodic and we are integrating over a period, we can replace the limits by \(-\pi\) to \(\pi\).

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin \left( (n+\frac{1}{2})\phi \right)}{\sin \left( \phi' \right)} f(\phi + \phi') d\phi
\]

If \( f(\theta) = 1 \)

\[
1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin \left( (n+\frac{1}{2})\phi \right)}{\sin \left( \phi' \right)} d\phi
\]

\[
\phi(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin \left( (n+\frac{1}{2})\phi \right)}{\sin \left( \phi' \right)} f(\phi + \phi') d\phi
\]

Subtraction

\[
f(\theta) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin \left( (n+\frac{1}{2})\phi \right)}{\sin \left( \phi' \right)} f(\phi + \phi') d\phi
\]

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin \left( (n+\frac{1}{2})\phi \right)}{\sin \left( \phi' \right)} \left( f(\phi + \theta) - f(\phi) \right) d\phi
\]
\[\frac{1}{2\pi} \int_0^\pi \frac{\sin((N+\frac{1}{2})\phi)}{\sin\left(\frac{\phi}{2}\right)} \left(f(\theta+\phi) - f(\theta)\right) + \frac{1}{2\pi} \int_0^\pi \frac{\sin((N+1)\phi)}{\sin\left(\frac{\phi}{2}\right)} \left(f(\theta - \phi) - f(\theta)\right) \]

\[= \frac{1}{2\pi} \int_0^\pi \frac{\sin((N+\frac{1}{2})\phi)}{\sin\left(\frac{\phi}{2}\right)} \left(f(\theta+\phi) + f(\theta - \pi) - 2f(\theta)\right) \]

The series approaches \( f(\theta) \) if the limit \( N \to \infty \) of this integral vanishes - there are many cases. How show this vanishes when \( f(x) \) is continuously differentiable.

**Fourier Transforms**

**Homework**

\[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + ipx} \, dx = e^{-p^2/2} \]

Take the complex conjugate

\[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} - ipx} \, dx = e^{-p^2/2} \]
If we use
\[ \frac{1}{\sqrt{2\pi}} \int \sum \text{Cn}(\text{Hn}(-i\frac{2}{dp}) e^{-p^2/2} - i p \cdot x) \, dp \]
we can integrate by parts
\[ \frac{1}{\sqrt{2\pi}} \int \sum e^{-p^2/2} \text{Cn} \text{Hn}(+i\frac{2}{dp}) e^{-i p \cdot x} \, dp = \]
\[ \sum \text{Cn} \text{Hn}(x) \frac{1}{\sqrt{2\pi}} \int e^{-p^2/2 - i p \cdot x} \, dp = \]
\[ \sum \text{Cn} \text{Hn}(x) e^{-x/2} \]
this shows that this translation can be inverted

the functions \( e^{-x^2/2} P_n(x) \) have the following properties:

1. \( x^k (e^{-x^2/2} P_n(x)) \to 0 \) as \( x \to \pm \infty \)

2. \( e^{-x^2/2} P_n(x) \) has an \( n \) number of derivatives

When these functions are Fourier transformed, then the transforms have an \( n \) \# of derivatives and satisfy \( p^k (e^{-p^2/2} P_n(p)) \to 0 \) as \( p \to \pm \infty \).
while the basis elements are elements of the Schwartz space, they are also members of $L^2(\mathbb{R})$ (the Hilbert space of square integrable functions).

The nice feature of $S$ is that the Fourier transform maps Schwartz functions to Schwartz functions. This is because powers $\leftrightarrow$ derivatives under Fourier transform.