Lecture 2

Last time we were discussing group representations:

\[ G = \text{group} \]

\[ D(g) : V \rightarrow V \]

\[ D(e) = I \]

\[ D(g_1)D(g_2) = D(g_1g_2) \]

\( V \) vector space, \( D(g) \), representation of \( G \)

Types of representations:

(1) regular representation

\[ \langle q \rangle \]

\[ \langle q'1q \rangle = s_{qq'} \]

\[ D(g)\langle q' \rangle = \langle q'g \rangle \]

In this case, the dimension of the vector space is the order of the group.
equivalent representations

If $S : V \rightarrow V$ has an inverse

$$D'(g) = S \, D(g) \, S'$$

defines an equivalent representation to $D(g)$.

* irreducible representation

$D(g)$ is irreducible if $D(g)$ has no non-trivial ($\neq 0, V$) subspaces in $V$.

* reducible representation

$P$ projection on a subspace of $V$ ($\neq 0, V$).

$$D(g)P = PD(g)P$$

* completely reducible representation

can be put in block diagonal form by similarity transormation, where each block is irreducible.
SD(g) S' = 
\[
\begin{pmatrix}
D_1(g) & 0 & \cdots & 0 \\
0 & D_2(g) & & \\
& & \ddots & \\
0 & & & D_m(g)
\end{pmatrix}
\]

where each \( D_i(g) \) is irreducible. This is called a direct sum \( SD(g) S' = D_1(g) \oplus D_2(g) \oplus \cdots \oplus D_m(g) \).

Examples:

Consider the regular representation of \( \mathbb{Z}_3 \) with similarity transformation

\[
D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

\[
D(a^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \omega = e^{2\pi i/3}
\]

\[
S = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad S' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}
\]

Recall \( \omega^3 - 1 = 0 = (\omega - 1)(1 + \omega + \omega^2) \)

since \( \omega \neq 1 \) \( 1 + \omega + \omega^2 = 0 \)

\[
S^{-1}D(e)S = D'(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
S^{-1}D(a)S = D'(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}
\]

\[
S^{-1}D(a^2)S = D'(a^2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & \omega \\ 0 & \omega & \omega^2 \end{pmatrix}
\]
we see that this fixed matrix $S$ diagonalizes the regular representation

\begin{align*}
D(e) &= D(a) = D(a^{-1}) = e \\
D(e) &= D(a) = \omega \quad D(a^{-1}) = \omega^2 \\
D(e) &= D(a) = \omega^2 \quad D(a^{-1}) = \omega
\end{align*}

The direct representation is the trivial representation, the other 2 are one-dimensional representations

\begin{align*}
D(e) &= 1 \quad D(a) = e \quad D(a^{-1}) = e \\
D(e) &= 4\pi i/3 \quad D(a) = e \quad D(a^{-1}) = e \\
D(e) &= 2\pi i/3 \quad D(a) = e \quad D(a^{-1}) = e
\end{align*}

Role of irreducible representations in quantum mechanics

\begin{align*}
[D(a), H] &= 0 \\
[S D(a) S^{-1}, SHS^{-1}] &= 0
\end{align*}

decompose into direct sum of irreducible representations

since $SHS^{-1}$ commutes with each $D_i(a)$, $H$ is also block diagonal in this basis $H \rightarrow H' = H_1 \oplus H_2 \cdots \oplus H_m$
diagonalizing $H$ = diagonalizing $H$ on each block.

Example 2: Space reflection

$p = \text{operation of space reflection}$

$p^2 = e$

This is a group with 2 elements.

The multiplication table is:

\[
\begin{array}{c|cc}
  & e & p \\
\hline
  e & e & e \\
  p & e & p \\
\end{array}
\]

In the regular representation:

\[
\begin{align*}
  D(e) |p\rangle &= |p\rangle & D(e) |e\rangle &= |e\rangle \\
  D(p) |p\rangle &= |p^2\rangle = |e\rangle & D(p) |e\rangle &= |pe\rangle = |p\rangle
\end{align*}
\]

In this basis:

\[
D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D(p) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

to diagonalize $|e\rangle$

\[
S = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad S = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]

\[
SD(e)S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
SD(p)S^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
This is block diagonal
\[ D_1(e) = D_1(p) = 1 \]
\[ D_2(e) = 1 \quad D_2(p) = -1 \]
If \([H, P] = 0\) we can find eigenstates of \( H \) that are symmetric or antisymmetric with respect to space reflection.

We can diagonalize \( H \) on the reflection symmetric or reflection antisymmetric subspace.

Example 2 \( S_3 \)

\[
D(c) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D(a_1) = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}
\]

\[
D(a_2) = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \quad D(p_{12}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
D(p_{23}) = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \quad D(p_{13}) = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}
\]

(because \( S_3 \) is a non-abelian group, some of the irreducible representations must be matrix representations.)
Not every reducible representation is completely reducible.

Example - consider the following matrix representation of addition of integers:

\[
\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} =
\begin{pmatrix} 1 & m+n \\ 0 & 1 \end{pmatrix}
\]

\[
D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(n^n) = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}
\]

Let \( P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) clearly

\[
D(n) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =
\begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
D(n) P = P = P D(n) P
\]

which means that \( D(g) \) is reducible.

Let \( Q = 1 - P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \)

\[
P D(n) Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq 0
\]
even though $P$ projects on an invariant subspace - $Q = 1 - P$ does not

* every representation of a finite group is equivalent to a unitary representation

(proof:

define $S = \sum_{g \in G} D^+(g) D(g)$

note

$S^+ = \sum_{g \in G} (D^+(g) D(g))^+ = \sum_{g \in G} D^+(g) D^+(g)$

$= \sum_{g \in G} D^+(g) D(g) = S$

so $S$ is a Hermitian operator

$\langle v | S | v \rangle = \sum_{g \in G} \langle v | D^+(g) D(g) | v \rangle = \sum_{g \in G} \langle v \mid D^+(g) D(g) | v \rangle = \langle v | D^+(g) D(g) | v \rangle + \sum_{g \neq e} \| D(g) | v \rangle \|^2$

$= \langle v | v \rangle + \sum_{g \neq e} \| D(g) | v \rangle \|^2 \geq \langle v | v \rangle > 0$

so $S$ is a strictly positive Hermitian operator
define
\[ D'(g) = S^{1/2} D(g) S^{-1/2} \]
\[ D'^*(g) = S^{-1/2} D^*(g) S^{1/2} \]
\[ D'^*(g) D(g) = S^{-1/2} D^*(g) S^{1/2} S D(g) S^{-1/2} = \]
\[ S^{-1/2} \sum_{q'} D^*(g') D(g') D(g) S^{-1/2} = \]
\[ D^*(g'') D(g'') \]

Let \( q'' = g' g \), \( q' = \frac{q}{q''} \), \( q'' = \frac{q}{q'} \)

\[ S^{1/2} S S^{-1/2} = I \]

which shows that \( D'(g) \) is a unitary representation of \( G \).

Theorem 2: Every reducible representation of a finite group is completely reducible.

From the previous theorem we can use a similarity transformation to transform \( D(g) \) to a unitary representation.
Let $D'(g)$ be a unitary representation and let $P$ project onto an invariant subspace. Invariance means

$$D(g)P = PD(g)P \quad \forall g \in G$$

Take adjoints

$$PD(g) = PD(g)P$$

Unitarily implies

$$D^+(g) = D(g) = D(g^{-1})$$

$$P D(g^{-1}) = P D(g)P \quad \forall g^{-1} \in G$$

Since $g^{-1}$ is an arbitrary element of $G$

$$PD(g) = PD(g)P$$

This means

$$D(g)(1-P) = D(g) - D(g)P =$$

$$D(g) - PD(g) = (1-P)D(g)$$

This means $D(g)$ is block diagonal in $P$, $1-P$. If $PD(g)P$ and $(1-P)D(g)(1-P)$ are irreducible we are done, otherwise at least one of them has a non trivial invariant subspace and the process can be
repeated until all blocks are irreducible, the completes the proof of the theorem

Return to more group theory

Definition:
A group \( H \) whose elements are elements of a group \( G \) is called a subgroup of \( G \)

Remarks
\( G \) is a subgroup of \( G \)
\( \{e\} \) is a subgroup of \( \{e\} \)

These are both trivial subgroups

Non-trivial examples
Let \( G = S_3 \)
\( \{e, e_1, e_2, e_3\} \) is a subgroup of \( S_3 \)
\( \{e, (1,2,3), (1,2,3)\} = \{e, a_1, a_2\} \) is a subgroup of \( S_3 \)
subgroup divide the group into equivalence classes

\[ q_1 \sim q_2 \text{ if } q_1 = h q_2 \]

the equivalence classes are called right cosets

If we denote \([g_i]\) = class containing \(g_i\)

\[ g_1 \sim g_2 \rightarrow g_1 = h q_2 \]

\[ g_2 = h' q_1 \]

\[ g_1 \sim q_2 \quad q_2 \sim q_3 \rightarrow g_1 \sim q_3 \]

\[ g_1 = h q_2 = h h' q_3 = h'' q_3 \quad h'' = h h' \]

we use the notation \( G/H \) space of cosets.

This is also possible to divide equivalence classes as

\[ q_1 \sim q_2 \text{ if } q_1 = g_2 h \]

where \(q_2\) is now on the left - these equivalence classes are called left cosets.
In general right and left cosets are not identical.

if \([g_7, r] = [g_c, 7] \quad hq = gh'

or

\[ gh q' = h' \]

Subgroups satisfying \( gh q' = H \) are called normal subgroups.

Examples: Let \( G = S_3 \), \( H = \langle P_{12} \rangle \)

\[
\begin{align*}
\text{e} & (1\ 2\ 3) = (1\ 2\ 3) = (2\ 3\ 1) = (1\ 2\ 3) \\
P_{12} & (1\ 2\ 3) = (1\ 2\ 3) = (2\ 3\ 1) = (1\ 2\ 3) \\
\text{e} & (2\ 3\ 1) = (2\ 3\ 1) = (1\ 2\ 3) = (2\ 3\ 1) \\
\text{e} & (1\ 2\ 3) = (1\ 2\ 3) = (1\ 2\ 3) = (1\ 2\ 3) \\
P_{12} & (2\ 3\ 1) = (2\ 3\ 1) = (1\ 2\ 3) = (1\ 2\ 3) \\
\text{e} & (1\ 3\ 2) = (1\ 3\ 2) = (1\ 3\ 2) = (1\ 3\ 2) \\
\text{e} & (2\ 3\ 1) = (2\ 3\ 1) = (1\ 2\ 3) = (1\ 2\ 3) \\
\text{e} & (1\ 3\ 2) = (1\ 3\ 2) = (1\ 3\ 2) = (1\ 3\ 2) \\
P_{12} & (1\ 3\ 2) = (1\ 3\ 2) = (1\ 3\ 2) = (1\ 3\ 2) \\
\text{e} & (1\ 2\ 3) = (1\ 2\ 3) = (1\ 2\ 3) = (1\ 2\ 3) \\
\text{e} & (2\ 3\ 1) = (2\ 3\ 1) = (1\ 2\ 3) = (1\ 2\ 3) \\
\text{e} & (1\ 3\ 2) = (1\ 3\ 2) = (1\ 3\ 2) = (1\ 3\ 2) \\
P_{12} & (1\ 3\ 2) = (1\ 3\ 2) = (1\ 3\ 2) = (1\ 3\ 2) \\
\text{e} & (1\ 2\ 3) = (1\ 2\ 3) = (1\ 2\ 3) = (1\ 2\ 3) \\
\text{e} & (2\ 3\ 1) = (2\ 3\ 1) = (1\ 2\ 3) = (1\ 2\ 3) \\
\text{e} & (1\ 3\ 2) = (1\ 3\ 2) = (1\ 3\ 2) = (1\ 3\ 2) \\
P_{12} & (1\ 3\ 2) = (1\ 3\ 2) = (1\ 3\ 2) = (1\ 3\ 2) \\
\end{align*}
\]

In this case the right coset containing \( (1\ 2\ 3) \) also has \( (1\ 2\ 3) \), while the left coset containing \( (1\ 2\ 3) \) also has \( (1\ 3\ 2) \). In this case the left and right cosets of \( G = S_3 \) with respect to \( \{ e, P_{12} \} \) are not the same.
on the other hand if

\[ H = \{ e, a_1, a_2 \} = \{ (1\ 2\ 3) \, (1\ 2\ 3) \, (2\ 3\ 1) \} \]

\[ hq = \begin{cases} \text{even permutation if } q \text{ is even} \\ \text{odd permutation if } q \text{ is odd} \end{cases} \]

we get the same set of cosets if we consider

\[ gh. \]

This means \( ghg^{-1} \) maps odd permutations to odd ones, even ones to even ones.

When \( H \) is a normal subgroup of \( G \) men the cosets form a group

\[ [e] = \text{coset containing identity} \]

\[ Hq \cdot Hq' = hq \cdot hq' = h(qh')q' = Hq'Hq' \]

\[ Hq \cdot Hq' = hqHq' = hq'Hq'\]

\[ = hh''qq' = Hq'q' \]

\[ Hq(Hq'Hq'') = Hq hh''qq'q'' = Hq h'h''qq'q'' = Hq'q'' \]

\[ (HqHq'Hq'')q'' = hh'qq'Hq'' = hh'qq'Hq'' \]

\[ = hh'h''qq'q'' \]
This shows that the cosets of a normal subgroup have the properties of a group.

We have just demonstrated that

\[ S_3/Z^3 \text{ is a group} \]

The group has 2 elements - odd and even permutations.

\[ S_3/Z^3 = \mathbb{Z}_2 \]

The center \( C \) of a group is the set of all elements of \( G \) that commute with every element of \( G \). Since

\[ e g = g e \]
\[ e g = g e = g \]
\[ g c^{-1} = c^{-1} g \]
\[ c_1 c_2 g = c_1 g c_2 = g c_1 c_2 \]

The center of a group is a normal subgroup of \( G \) - it may be trivial (just \( e \), or it could be the whole group - in general it is an abelian normal subgroup.)
Let $G$ act on a set $S$, we can define an equivalence relation on $S$ by

$$s' \sim s \text{ if } s' = g^{-1}sg$$

In some $g$, these classes are called \textit{conjugacy classes}.

$$s_1 \sim s_2 \quad s_1 = g^{-1}s_2g \quad s_2 = g^{-1}s_1g \quad s_2 \sim s_1$$

$$s_1 \sim s_1 \quad s_1 = g^{-1}s_1g = g^{-1}s_2g = g^{-1}s_3g = \cdots$$

These are sets - not subgroups - there is no assumption about multiplication when $G$ is a group the mappings

$$G \to G' = gGg^{-1} \quad g \text{ fixed } \in G$$

is called an \textit{inner automorphism} - it clearly preserves all of the group properties.

An \textit{outer automorphism} is a mapping from $G \to G$ that preserves the group structure that cannot be written as $G' = gGg^{-1}$ for any element of $G$. 
Now we come to one of the more important results of representation theory.

**Schur's Lemma**

**Part 1:** Assume that $D_1(g)$ and $D_2(g)$ are inequivalent irreducible representations of $G$, and $A$ is an operator satisfying

$$D_1(g)A = AD_2(g) \quad \forall g \in G$$

then $A = 0$

**Part 2** Assume $D_1(g)$ is an irreducible representation of $G$ and $A$ is an operator satisfying

$$D_1(g)A = AD_1(g) \quad \forall g \in G$$

then $A$ is a constant multiple of the identity.
Assume that there is a vector $w$ satisfying

$$Aw = 0$$

It follows that

$$D(q)w = AD(q)w = 0 \quad \forall q \in G$$

This means that the subspace spanned by $D(q)w$ is an invariant subspace of $D(q)$, but $D(q)$ is irreducible, so it is the whole space.

$$\therefore A = 0$$

This means $A$ has no null space, if $A$ is not $0$.

Assume there is a vector $w$ satisfying

$$\langle w | A = 0 = 1 \quad \langle w | AD(q) = 0 \quad \forall q \in G$$

$$\langle w | D(q) = 0 \quad \forall q \in G$$

For the same reason as before, $\langle w | D(q)$ generates an invariant subspace, since $D(q)$ is irreducible.
This means \( \langle v | A = 0 \rangle \) for all vectors on \( A \) is 0. If \( A \) is not 0 then

(1) \( A | v \rangle \neq 0 \) \( \langle w | A = 0 \rangle \)

for any vector. The first equation means

\[
| v \rangle \neq | w \rangle \quad \Rightarrow \quad A | v \rangle \neq A | w \rangle \\
A( | v \rangle - | w \rangle ) = 0
\]

so \( A \) is 1–1.

Similarly if there is a non-trivial vector satisfying \( \langle w | A = 0 \rangle \) this means \( A \) is not onto. Since \( A \neq 0 \), this cannot happen.

\( \therefore A \) is 1–1 and onto. Since these are finite-dimensional vector spaces, \( A \) has an inverse so the dimensions of both spaces are the same. This means

\[
D_{2}(5) A = A D_{1}(5) \\
A^{\dagger} D_{2}(5) A = D_{1}(5)
\]

which contradicts the assumption that these representations are equivalent.
For part 2 assume
\[ D_i(s) A = AD_i(s) \]
\[ D_i(s) (A - \lambda I) = (A - \lambda I) D_i(s) \quad \forall \lambda \in \mathbb{C} \]
\[ \det (A - \lambda I) \text{ is a polynomial in } \lambda \]
It has at least 1 root (even when we have generalized eigenvectors—there are also eigenvectors)

Let \[ A \mathbf{v} = \lambda \mathbf{v} \]

\[ D_i(s) (A - \lambda I) \mathbf{v} = 0 = (A - \lambda I) D_i(s) \mathbf{v} \]

by irreducibility \[ D_i(s) \mathbf{v} \] defines an invariant subspace, it must be the whole space \( \Rightarrow \) the means
\[ A = \lambda I \]

which completes the proof of Schur's lemma.
Application of Schur's Lemmas.

* Let $\Omega$ be an operator that is invariant under $D(q)$,

$$[\Omega, D(q)] = 0 \quad \forall q \in G$$

Assume that $D(q)$ is completely reducible, let $S$ be the similarity transformation that block diagonalizes $D(q)$.

This defines a change of basis

$$|n y m\rangle$$

$$\langle n y m | D(q) | n' y' m' \rangle =$$

$$S_{n n'} S_{y y'} D_{m m', (q)}^{y y'}$$

is irreducible

$I = \sum |n y m\rangle \langle n y m|$,

$D(q) = \sum |n y m\rangle \langle n y m | D(s) | n' y' m' \rangle \langle n' y' m'|$

$$= \sum |n y m\rangle D_{m m', (s)}^{y y'} \langle n y m'|$$
Consider \( D(\phi)_\lambda \) in this basis

\[
0 = \sum \langle \eta \gamma m' | a \Omega | \eta' \gamma' m'' \rangle \tilde{D}^\Omega_{m''m'}(\phi) \langle \eta' \gamma' m'' | a \Omega | \eta \gamma m' \rangle
\]

in terms of the matrices:

\[
0 \tilde{D}^\phi = \tilde{D}^\phi 0
\]

By Schur's lemma, this means that \( 0 \) is 0 unless \( \phi = \phi' \)

\[
\langle \eta \gamma m | a \Omega | \eta' \gamma' m'' \rangle \tilde{D}^\Omega_{m''m'}(\phi) = \tilde{D}^\phi_{m m'}(\phi') \langle \eta' \gamma' m'' | a \Omega | \eta \gamma m' \rangle
\]

\( \forall \eta \in \mathfrak{g} \)

By the second part of this lemma,

\[
\langle \eta \gamma m | a \Omega | \eta' \gamma' m' \rangle = \omega^\phi_{m m'} \delta_{\eta \eta'} \delta_{\gamma \gamma'} \delta_{m m'}
\]

which means that all of the dependence of \( \Omega \) that does not follow from group theory is in the coefficients \( \omega^\phi_{m m'} \).