Lecture 22

compact operators and integral equations.

A is compact if \( \forall \varepsilon > 0 \) there is a finite dimensional operator \( F \) satisfying

\[ \| A - F \| < \varepsilon \]

These operators are the closest operators to finite dimensional matrices.

* To show an operator is compact

\[
\text{Tr}(A^*A) < \infty \\
(\text{Tr}((A^*A)^p))^{\frac{1}{p}} < \infty
\]

(all sufficient conditions for compactness)

* Canonical sum

\[ A = \sum \lambda_n \psi_n \psi_n^* \]

\[ \langle \psi_n | \psi_m \rangle = \delta_{mn} \]

\[ \langle \psi | \psi_m \rangle = \delta_{mn} \]

\[ \lambda_n \geq 0 \rightarrow 0 \text{ as } n \rightarrow \infty \]
structure of equations

\[ f(x) = q(x) + \int_{a}^{b} A(xy) f(y) dy \]

* check compactness

\[ \int |A(x,y)|^2 \, dx \, dy < \infty \]

* solution by iteration - first

find \( F = \sum_{k=1}^{K} v_{kn}(x) \lambda_{mn} v_{kn}(y) \)

\[ \|A - F\| < \frac{1}{2} \]

Abstractly

\[ (1 - F) | \psi \rangle = | \psi \rangle + (A - F) | \psi \rangle \]

\[ | \psi \rangle = (1 - F)^{-1} | \psi \rangle + (1 - F)^{-1} (A - F) | \psi \rangle \]

What do we mean by \( 1 - F \)

Assume \( \langle n_{m} l_{m} | n_{m} l_{m} \rangle = \delta_{nm} \)

\[ 1 = (1 - F)(1 + F) \quad (1 + F) = (1 - F)^{-1} \]

\[ 1 = 1 - \Sigma \langle l_{m} | (\lambda_{nm} - q_{nm} + \Lambda_{LR} q_{km}) | l_{m} \rangle \]

\[ (\delta_{nm} - \lambda_{nm}) q_{mn} = \lambda_{n} \]

\[ q_{mn} = (\delta_{nm} - \lambda_{nm})_{mn} \]

\[ G_{mn} = (\delta_{nm} - \lambda_{nm})_{mn} \]

\[ = \lambda_{n} \]
\[ f(x) = (1+\delta)q + (1+\delta)(A-F)f \]
\[ = q(x) + \sum h_n(x) g_{nm} \langle h_m | q \rangle + \int (A-F)(x,y) + \sum h_n(x) g_{nm} h_m (y) (A-F)(z) \rangle \langle f(y) | q \rangle \]

This is the sum of the integral equation.

The important point is

\[ f = (1+\delta)q + (1+\delta)(A-F)q \]
\[ = \sum_{n=0}^{\infty} [(1+\delta)(A-F)]^n (1+\delta)q \]

This implies convergence.

Solution by Galerkin method.

Choose any orthonormal basis

\[ \langle n_1 f \rangle = \langle n_1 q \rangle + \sum \langle n_1 k, l m \rangle \langle m l f \rangle \]

For a large enough \( N \), \( PKP \to K \)

In norm

\[ \langle n_1 f \rangle = \sum (S_{nk} - \langle n_1 k | f \rangle)_{nm} \langle m l q \rangle \]

Finite matrix inversion.
\[ \langle x_{15} \rangle = \langle x_{19} \rangle + \sum_m K(x_m) \langle m|15 \rangle \]

This method requires computing \( \langle n_{19} \rangle \) \( \langle n_{19} | m \rangle \). Its accuracy can be improved with a good choice of basis.

\[
\begin{align*}
|n_1\rangle &= |n_1\rangle \\
A|n_1\rangle &= |n_2\rangle \\
A^2|n_1\rangle &= |n_3\rangle \\
A^n|n_1\rangle &= |n_n\rangle
\end{align*}
\]

du a hermitian A

\[
A = \sum |n\rangle \langle n| \\
A^n = \sum |n\rangle \langle n| \\
\text{biggest contribution from largest eigenvalues}
\]

This, when used with the Galerkin method, is called the Gram–Schmidt method.
collocation method.

* Choose point and weights on $\Sigma$ oh $\Sigma$

$x = \alpha y + \beta$

$y = -1 \quad x = a \quad a = -\alpha + \beta$

$y = 1 \quad x = b \quad b = \alpha + \beta$

$\beta = \frac{1}{2}(a + b) \quad \alpha = \frac{b - q}{2}$

$x = \frac{b - q}{2} y + \frac{a + b}{2}$

d$x = \frac{b - q}{2} dy$

$x_n = \frac{b - q}{2} y_n + \frac{a + b}{2}$

$d x_n = \frac{b - q}{2} \delta n$

$\langle x_n 1 S \rangle = \langle x_n 1 q \rangle + \sum K(x_n y_n) \omega_n \langle y_n 1 S \rangle$

$\sum (\delta_{nm} - K(x_n y_m) \omega_m) \langle x_m 1 S \rangle = \langle x_n 1 q \rangle$

Solving gives

$\langle x_m 1 S \rangle = \sum (S - K\omega)^{m n} \langle x_n 1 q \rangle$

$\langle x 1 S \rangle = \langle x 1 q \rangle + \sum K(x, x_n) \omega_n \langle x_n 1 S \rangle$
Volterra integral equations

\[ \frac{\partial \Psi}{\partial t} = A(t) \Psi \]

we write this as an integral equation

\[ |\Psi(t)\rangle = |\Psi(0)\rangle + \int_0^t A(t') |\Psi(t')\rangle dt' \]

we can try to solve this by iteration

\[ |\Psi(t)\rangle = |\Psi(0)\rangle + \int_0^t A(t') |\Psi(t')\rangle dt' \]

\[ \int_0^t A(t') \int_0^{t'} A(t'') |\Psi(t'')\rangle dt'' dt' + \cdots \]

relabel

\[ \int_0^t A(t') dt'' \int_t^\infty A(t') dt' |\Psi(t)\rangle \]
we write this as
\[ \frac{1}{2} \int_0^t dt \int_0^t dt' \ T(\text{Alt}_t \text{Alt}'_t) \]

\[ T(\text{Alt}_t \text{Alt}'_t) = \Theta(t-t') \text{Alt}_t \text{Alt}'_t + \Theta(t'-t) \text{Alt}'_t \text{Alt}_t \]

when there are 3 As there are 3; order of integration

\[ \frac{1}{3!} \int_0^t dt_1 \int_0^t dt_2 \int_0^t dt_3 \ T(\text{Alt}_{t_1} \text{Alt}_{t_2} \text{Alt}_{t_3}) \]

This gives
\[ |\Psi(t)\rangle = \sum_{n=0}^{\alpha} \frac{1}{n!} \int_0^t dt_1 \cdots dt_n \ T(\text{Alt}_{t_1} \cdots \text{Alt}_{t_n}) |\Psi(0)\rangle \]

If A is bounded, it is uniformly bounded on $[0, t]$

\[ |||A|||_m = \sup_{0 \leq t \leq t} |||A(t)||| \]

\[ |||T(\text{Alt}_{t_1} \cdots \text{Alt}_{t_n})||| \leq |||A|||_m^n \]

The terms in the sum are bounded by
\[ \frac{1}{n!} t^n |||A|||_m^n \]
This means that the infinite sum converges - it is bounded by $e^{\|A\|}$. 

where $\|A\| = \sup_{t \in \mathbb{T}} \|A(t)\|$. 

This way of solving the equation is especially useful if $[A(t), A(t')] \neq 0$. 

This type of integral equation is called a Volterra integral equation. 

The series solution in terms of time ordered products is called the Dyson series.
An ordinary differential equation of N order is an equation of the form

$$F(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}) = 0$$

The differential equation is linear if F is a linear function of y and its derivatives.

$$\sum_{n=0}^{N} C_n(x) \frac{d^n y}{dx^n} + g(x) = 0$$

We are mostly interested in the case that $C_n(x) \neq 0$ so it can be divided out.

The equation is homogeneous if $g(x) = 0$; otherwise it is inhomogeneous.

In general, it is important to specify the class of solutions (differentiable, distributional, etc.)

It is also necessary to specify boundary conditions - $y$, $y'$, $y''$, etc.

In example.
consider one special case
\[ \frac{d^n f}{dx^n} = F(x, f, \frac{df}{dx}, \frac{d^n f}{dx^n}) \]

we can express this as the system
\[ \frac{d q_{n-1}}{dx} = F(x, f, q_1, q_{n-1}) \]
\[ \frac{d q_{n-2}}{dx} = q_{n-1} \]
\[ \frac{d q_{n-3}}{dx} = q_{n-2} \]
\[ \frac{d q_1}{dx} = q_2 \]
\[ \frac{d f}{dx} = q_1 \quad (\text{call } f = q_0) \]

this is a system of \( n \) first order equations
\[ \frac{d q_k}{dx} = G(x, q) \]

integrating
\[ q_k(x) = q_k(0) + \int_0^x G_k(x', q) \, dx' \]

there are the values of \( f \)
and its first \( n-1 \) derivative at 0
Cauchy and Lipschitz give conditions that ensure \( G(x, \xi) \) satisfies bounds that make the iterative solution of this equation convergent.

The beauty of this result is that it also holds for non-linear equations.

In general, initial conditions are not the only kind of boundary conditions that arise.

In what follows we limit considerations to 2nd order equations:

\[
\alpha(x) \frac{d^2 u}{dx^2} + b(x) \frac{du}{dv} + c(x) u(v) = f(v)
\]

\[
B_1 = \alpha_1 U(a) + \beta_1 \frac{du}{dx}(a) + \gamma_1 U(b) + \delta_1 \frac{du}{dx}(b) = s_1
\]

\[
B_2 = \alpha_2 U(a) + \beta_2 \frac{du}{dx}(a) + \gamma_2 U(b) + \delta_2 \frac{du}{dx}(b) = s_2
\]

We take constants are chosen so \( B_1 \) and \( B_2 \) are linearly dependent.
If \( \sigma_1 = \sigma_2 = 0 \) the boundary conditions are called homogeneous boundary conditions; if any of the \( \sigma_i \neq 0 \) then the boundary conditions are called inhomogeneous.

*First order equations*

\[
\frac{a(x)}{dx} \frac{du}{dx} + b(x)u(x) = f(x) \quad a(x) \neq 0
\]

Let \( \frac{b}{a} = \frac{1}{p(x)} \frac{dp}{dx} \)

\[
\frac{du}{dx} + \frac{b}{a} u = \frac{1}{a} f
\]

\[
\frac{du}{dx} + \frac{1}{p} \frac{dp}{dx} u = \frac{1}{a} f
\]

\[
p \frac{du}{dx} + \frac{dp}{dx} u = \frac{p}{a} f
\]

\[
\frac{dp}{dx} (pu) = \frac{p}{a} f
\]

\[
p \ u = \int_{x_0}^{x} \frac{p}{a} f \quad p(v) = e^{\int_{a(x')}^{v} \frac{b}{a(x')} dx'}
\]

\[
u(x) = \frac{1}{p(x)} \left[ \int_{x_0}^{x} \frac{p(x')}{a(x')} f(x') dx' + u(x_0) \right]
\]

\[(p(x_c) = 1)\]

In this case the solution is fixed by \( u(x_c) \).
next consider

\[ au'' + bu' + cu = g \]

and

\[ au'' + bu' + cu = 0 \]

Let \( u_1 \) and \( u_2 \) be 2 solutions of the homogeneous equation

\[ au_1 + bu_2 \]

is also a solution for any constants \( \alpha, \beta \).

Linear independence of \( u_1, u_2 \):

\[ \alpha u_1 + \beta u_2 = 0 \Rightarrow \alpha = \beta = 0 \]

Thus

\[ \alpha u_1 + \beta u_2 = 0 \]
\[ \alpha u_1' + \beta u_2' = 0 \]

\[ \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \]

The criterion for independence of \( u_1 \)

\[ W = \det \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} = \text{Wronskian} \neq 0 \]
If \( W = 0 \), \( u_1 u_2' = u_2 u_1' \) then

\[
\frac{u_1'}{u_1} = \frac{u_2'}{u_2} \quad \ln u_1 = \ln u_2
\]

\( u_1 = \text{const} \times u_2 \)

If there were 3 independent solutions,

\[
\det \begin{pmatrix} u_1 & u_2 & u_3 \\ u_1' & u_2' & u_3' \\ u_1'' & u_2'' & u_3'' \end{pmatrix} \neq 0
\]

linear combinations
of \( u, u' \)

This must vanish because the 2nd derivatives are not independent of the 0th and first derivatives.

A second order homogeneous differential equation can have at most 2 independent solutions - satisfying \( W(u, u_1) \neq 0 \)

Let \( u = u_p + v \)

Let

\[
a u'' + bu' + cu = q
\]
Then $v$ is a solution of the homogeneous equation which means that the general solution has the form

$$v(x) = v_0(x) + c_1 u_1(x) + c_2 u_2(x)$$

where $c_1, c_2$ are fixed by boundary conditions.

Theorem

Assume $u_1(x)$ is a solution of

$$a u'' + bu' + cu = 0$$

then we can always find a second solution of the form

$$u_2(x) = u_1(x) h(x)$$

$$u_2' = u_1' h(x) + u_1 h'$$

$$u_2'' = u_1'' h(x) + 2 u_1' h' + u_1 h''$$

$$a u_2'' + b u_2' + c u_2 = 0$$

$$a u_1'' + b u_1' + c u_1 = 0$$

$$0 = b u_1 h' + a (2 u_1 h' + u_1 h'')$$

$$(b u_1 + 2 a u_1') h' = -a u_1 h''$$
This gives

\[ \frac{h''}{h'} = \frac{bu_1 + 2au_1}{au_1} = p = \frac{b}{a} \]

\[ \ln h' = -\int p(x) - 2\ln u_1 \]

\[ h' = e^{-\int^x (p(x'))dx} / u_1^2 \]

\[ u_2(x) = u_1(x) \int_{x_0}^x \frac{dx''}{u_1(x'')} e^{-\int_{x_0}^{x''} p(x''')dx'''} \]

Since

\[ u_2 = hu_1 \]
\[ u_2' = h'u_1 + hu_1' \]

\[ \begin{pmatrix} u_1 \\ h'u_1 + hu_1' \end{pmatrix} = \begin{pmatrix} u_1' \\ h'u_1 + hu_1' \end{pmatrix} = n'u_1 + nu_1' = hu_1' \]

\[ h'u_1^2 = W = \int_{x_0}^x e^{-\int_{x_0}^{x'} p(x'')} dx' \]