Starting next Monday I will be giving lectures on panopto via ICON. There are 15 lectures left - normally two would be used for review and one for a mid-term exam. This leaves 12 lectures. My intention is to break each lecture up into 3-4 topics. I will post a latex file for each topic on the regular class web page and in the video I will go discuss the latex file line by line. You can make a copy of the latex file if you want to write notes. If there is any algebra involved it would be a good idea to work through it on your own. The videos will be short clips - dealing with one topic at a time. The total time for 3-4 videos will take less than the 50 minutes since I will not be writing the blackboard. I will try to pace myself so we cover most of what is in the text during the 13 weeks of lecture. As far as exams go I will give take home exams for the last mid term and final. Please feel free to e-mail me with any questions. I have put together a mailing list so it might be easiest if you "forward all" any questions so everyone on the list can see them and comment. All of this is new to me - so please contact me by-email if you have any problems.

Three lectures are already posted on Icon. To go to them you have to go to modules → pages → show all pages. The text files will all be posted in the class web page. There is a running lecture that has them all.

Stay healthy - I look forward to seeing all of you in person when this crisis is over!
Wayne

29:4762 Lecture 23 - Monday - March 23

In this lecture I

• Finish the construction of a second solution of the homogeneous equation given a first solution.
• Construct the solution of the inhomogeneous differential equation given a solution of the homogeneous equation.
• Introduced adjoint differential operators and adjoint boundary conditions.
• Self-adjoint operators
• Green functions

Lecture 23 - Topic 1: Constructing a second solution to the homogeneous equation from a first solution

Let $u_1(x)$ be a solution of the homogeneous differential equation:

$$a(x)u''_1(x) + b(x)u'_1(x) + c(x)u_1(x) = 0.$$ 

Look for another solution of the form $u_2(x) = h(x)u_1(x)$. Requiring $u_2(x)$ to satisfy the differential equation gives

$$h(x)(a(x)u''_1(x) + b(x)u'_1(x) + c(x)u_1(x)) +$$
\[ h'(x)(2a(x)u_1'(x) + b(x)u_1(x)) + h''(x)a(x)u_1(x) = 0. \]
The coefficient of \( h(x) \) vanishes because \( u_1(x) \) is a solution of the differential equation. This gives the following equation for \( h'(x) \)
\[
\frac{h''(x)}{h'(x)} = -\frac{1}{u_1(x)a(x)} (2a(x)u_1'(x) + b(x)u_1(x)) = -2\frac{u_1'(x)}{u_1(x)} - \frac{b(x)}{a(x)}.
\]
In order to integrate this equation define \( p(x) \) by
\[
\frac{1}{p} \frac{dp}{dx} = \frac{b(x)}{a(x)}.
\]
This can be integrated to find \( p(x) \):
\[
p(x) = e^{\int_{x_0}^{x} (b(x')/a(x'))dx'}.\]
The equation for \( h'(x) \) in terms of \( p(x) \) becomes
\[
\frac{h''(x)}{h'(x)} = -2\frac{u_1'(x)}{u_1(x)} - \frac{p'(x)}{p(x)}.
\]
Integrating this
\[
\ln(h'(x)) = -2\ln(u_1(x)) - \ln(p(x)) + c.
\]
Taking exponents gives
\[
h'(x) = \frac{1}{u_1(x)^2 p(x)} = \frac{e^{\int_{x_0}^{x} (b(x')/a(x'))dx'}}{u_1(x)^2}.
\]
where we have used the expression for \( p(x) \). Integrating once again gives
\[
h(x) = \int_{x_0}^{x} dx' e^{\int_{x_0}^{x} (b(x'')/a(x''))dx''} \frac{1}{u_1(x')^2}
\]
and an expression for the other solution:
\[
u_2(x) = u_1(x)h(x) = u_1(x) \int_{x_0}^{x} dx' e^{\int_{x_0}^{x} (b(x'')/a(x''))dx''} \frac{1}{u_1(x')^2}
\]
The Wronskian for this pair of solutions is
\[
W(x) = \begin{vmatrix} u_1(x) & u_1(x)h(x) \\ u_1'(x) & u_1'(x)h(x) + h'(x)u_1(x) \end{vmatrix} = u_1^2(x)h'(x) = \frac{1}{p(x)} = e^{\int_{x_0}^{x} (b(x')/a(x'))dx'} > 0.
\]
Since this is never 0 these solutions are linearly independent.

Remark: all of the classical orthogonal polynomials are solutions of second order differential equations. The second solution normally is not a polynomial. This method can be used to construct the non-polynomial solutions from the polynomial solutions.
Lecture 23 - Topic 2: Constructing a solution to the inhomogeneous equation from a solution of the homogeneous equation

Let $u_1(x)$ be a solution of the homogeneous equation:

$$a(x)u''_1(x) + b(x)u'_1(x) + c(x)u_1(x) = 0.$$ 

Look for a solution of the inhomogeneous equation

$$a(x)u''_f(x) + b(x)u'_f(x) + c(x)u_f(x) = f(x)$$

of the form $u_f(x) = v(x)u_1(x)$. Requiring $u_f(x)$ to satisfy the inhomogeneous differential equation gives

$$v(x)(a(x)u''_1(x) + b(x)u'_1(x) + c(x)u_1(x)) +$$

$$v'(x)(2a(x)u'_1(x) + b(x)u_1(x)) + v''(x)a(x)u_1(x) = f(x).$$

As in the homogeneous case, the coefficient of $v(x)$ vanishes because $u_1(x)$ is a solution of the homogeneous differential equation. This gives the following equation for $v'(x)$

$$v''(x) = -\frac{v'(x)}{a(x)u_1(x)} (2a(x)u'_1(x) + b(x)u_1(x)) + \frac{f(x)}{(u_1(x)a(x))} =$$

$$-v'(x) \left( 2 \frac{u'_1(x)}{u_1(x)} + \frac{b(x)}{a(x)} \right) + \frac{f(x)}{(u_1(x)a(x))}.$$ 

In this case we define

$$p(x) = \frac{b(x)}{a(x)}$$

and $R(x)$ by

$$\frac{1}{R(x)} \frac{dR}{dx} = \left( 2 \frac{u'_1(x)}{u_1(x)} + \frac{b(x)}{a(x)} \right)$$ \hspace{1cm} (2)$$

This choice of $R(x)$ can be used to integrate the equation.

$$v''(x) + v'(x)R'(x)/R(x) = \frac{f(x)}{a(x)u_1(x)}$$

$$\frac{d}{dx}(v'(x)R(x)) = \frac{f(x)R(x)}{a(x)u_1(x)}.$$ 

Integrating this equation gives

$$v'(x) = \frac{1}{R(x)} \int_{x_0}^x dx' \frac{f(x')R(x')}{a(x')u_1(x')}$$

where $R(x)$ is constructed by integrating (2), which gives

$$R(x) = u_1^2(x)e^{\int_{x_0}^x \frac{b(x')}{a(x')}} dx'.$$
Using this in the expression for $v$ and $\mu$

This can be made a little prettier if we use the expression for the Wronskian (1)

Integrating and multiplying by $u$ gives

One more integration gives $v(x)$ and $u_f(x)$:

and

This can be made a little prettier if we use the expression for the Wronskian (1)

where $h(x)$ is the $h(x)$ form the previous topic. Using this $R^{-1}(x)$ can be expressed as

Using this in the expression for $v'(x)$ gives

Integrating and multiplying by $u_1(x)$ gives an expression for the solution of the inhomogeneous equation in terms of both solutions of the homogeneous equation:

These results show that having one solution of homogeneous equation is sufficient to get the second solution to the homogeneous equation and a solution of the inhomogeneous equation by integration. To use these methods it is necessary to have a solution of the homogeneous equation. In addition this method is cannot be simply generalized to treat higher order equations or partial differential equations.

Lecture 23 - Topic 3: The generalized Green’s identity and adjoint operators
Consider a second order ordinary differential operator of the form

\[ L_x := a_2(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_0(x) \]

You should think of \( L_x \) acting on a space of functions, which is called the domain of \( L_x \).

Let \( w(x) > 0 \) be a positive weight function on the interval \([a, b]\).

Consider the difference

\[ w(x)(v^*(x)L_xu(x) - u(x)(L_x^\dagger)v(x))^* \]

where \( L_x^\dagger \) is an operator that has not yet been defined.

Assume that there is an \( F(u, v^*) \), of the general form

\[ F[u, v^*] = A(x)v^*(x)u(x) + B(x)v^*(x) \frac{du(x)}{dx} + C(x) \frac{dv^*(x)}{dx} u(x) + D(x) \frac{dv^*(x)}{dx} du(x), \]

i.e. it is bilinear in \( u, v^* \) and their derivatives, that satisfies

\[ w(x)(v^*(x)L_xu(x) - u(x)(L_x^\dagger)v(x))^* = \frac{dF(u, v^*)}{dx}. \]  \( (3) \)

Equation (3) is called Lagrange’s identity. This identity defines \( L_x^\dagger \) when the right side is a total derivative. The operator \( L_x^\dagger \) is called the formal adjoint of \( L_x \). This can be integrated from \( a \) to \( b \) to get

\[ \int_a^b dxw(x)(v^*(x)L_xu(x) - u(x)(L_x^\dagger)v(x))^*) = F(u(b), v^*(b)) - F(u(a), v^*(a)) \]  \( (4) \)

Equation (4) is called the Generalized Green’s identity. The term on the right side is called the boundary term. When \( L_x = L_x^\dagger \), \( L_x \) is called a self-adjoint operator.

Lecture 23 - Topic 4: Adjoint boundary conditions, vector spaces

We begin by considering the case where the boundary condition are homogeneous:

\[ A_1u(a) + B_1u'(a) + C_1u(b) + D_1u'(b) = 0 \]
\[ A_2u(a) + B_2u'(a) + C_2u(b) + D_2u'(b) = 0 \]

In this case functions satisfying these boundary conditions form a vector space \( U \) since sums and scalar multiples of functions satisfying these conditions form a linear vector space (this would not be true with inhomogeneous boundary conditions).
We look for boundary conditions of the functions \( v(x) \) that make the boundary terms in the generalized Green’s identity vanish. We start by writing

\[
0 = F[u, v^*](b) - F[u, v^*](a) = \alpha_2(b)v^*(b)u(b) + \beta_2(b)v^*(b)u'(b) + \gamma_2(b)v''(b)u(b) + \delta_2(b)v''(b)u'(b)
- \alpha_1(a)v^*(a)u(a) - \beta_1(a)v^*(a)u'(a) - \gamma_1(a)v''(a)u(a) - \delta_1(a)v''(a)u'(a).
\]

(5)

The boundary condition on \( u(x) \) can be used to eliminate two of the four quantities \( u(a), u'(a), u(b), u'(b) \). This means that the coefficients of the remaining two quantities are each linear combinations of \( v(a), v'(a), v(b), v'(b) \). Setting these coefficients to zero gives a pair of homogeneous boundary conditions on the functions \( v \). These are called the adjoint boundary conditions. Functions satisfying these conditions form another vector space \( V \).

It is instructive to consider an example. Assume the boundary conditions on \( u(x) \) are \( u(a) = u(b) = 0 \). Using these conditions in (5) gives

\[
0 = F[u, v^*](b) - F[u, v^*](a) = u'(b)(\beta_1(b)v^*(b) + \delta_2(b)v''(b)) - u'(a)(\beta_1(a)v^*(a) - \delta_1(a)v''(a)).
\]

This will vanish if

\[
\beta_2(b)v^*(b) + \delta_2(b)v''(b) = 0
\]

and

\[
\beta_1(a)v^*(a) + \delta_1(a)v''(a) = 0
\]

These are the adjoint boundary conditions. They are designed to make the boundary terms vanish, once the boundary conditions are applied.

The functions \( v(x) \) with the adjoint boundary conditions also form a linear vector space, \( V \).

If we consider functions \( u(x) \) satisfying boundary conditions and \( v(x) \) satisfying the adjoint boundary conditions it follows that

\[
\int_a^b dxw(x)(v^*(x)L_xu(x) = \int_a^b dxw(x)u(x)(L_x^\dagger)v(x))^*
\]

(6)

This relation is called Green’s identity. This can be expressed in the notation

\[
\langle v\|L_x\|u \rangle = \langle u\|L_x^\dagger\|v \rangle^*
\]

While this has the same form as the corresponding equation for matrices, in the case of differential operators there are important restrictions on the boundary conditions for this relation to hold.

It is important to note that the form of \( L_x \) does not define the operators - the boundary conditions that determine the domain are also considered as part of the definition of the operator.

An differential operator \( L_x \) is self-adjoint if

\[
L_x = L_x^\dagger
\]

and the vector spaces \( U \) and \( V \) are identical - this means that the boundary conditions of \( u(x) \) are identical to the adjoint boundary conditions on \( v(x) \).
In this lecture I discuss
• Self-adjoint operators
• Green’s functions
• Self-adjoint Green’s functions
• Computing Green’s functions

Lecture 24 - Topic 1: Self-adjoint operators

Consider a second order differential operator of the form

\[ L_x = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} + c(x) \]

where in general the coefficients may be complex, with the real part of \( a(x) \) positive. Consider the case where \( w(x) = 1 \).

The standard way to construct the adjoint operator is to integrate by parts to move the derivatives from \( u(x) \) to \( v^*(x) \).

\[
\langle v|L_x|u \rangle = \int_a^b dx v^*(x)(a(x)u''(x) + b(x)u'(x) + c(x)u(x)) =
\]

\[
\int_a^b dx (-v^*(x)a(x))'u'(x) - (v^*(x)b(x))'u(x) + v^*c(x)u(x)) +
\]

\[ [v^*a(x)u'(x) + v^*(x)b(x)u(x)]|^b_a =
\]

\[
\int_a^b dx ((v^*(x)a(x))'' - (v^*(x)b(x))' + v^*c(x))) u(x) +
\]

\[ [v^*a(x)u'(x) + v^*(x)b(x)u(x) - (v^*(x)a(x))'u(x)]|^b_a
\]

\[ \langle L^\dagger v|u \rangle + [v^*a(x)u'(x) + v^*(x)b(x)u(x) - (v^*(x)a(x))'u(x)]|^b_a
\]

We can read off the form of the adjoint differential operator

\[ L^\dagger = a^* \frac{d^2}{dx^2} + (2a^* - b^*) \frac{d}{dx} + (a''^* - b'^* + c^*) \]

Comparing this to the expression of \( L_x \) we see that self-adjointness, \( L_x = L_x^\dagger \) requires

\[ a(x) = a^*(x); b(x) = 2a(x)' - b^*(x); c(x) = a''(x) - b''(x) + c^*(x) \]

The solution is \( a(x) = a^*(x), b(x) = a'(x), c(x) = c^*(x) \). With these choices

\[ L_x = L_x^\dagger = \frac{d}{dx}(a(x) \frac{d}{dx}) + c(x) \]
and the boundary terms become

\[ a(x)[v^*(x)u'(x) - v''(x)u(x)]]_a^b. \]

Next we show that all second-order differential operators with real coefficients are self-adjoint, if a suitable positive weight \( w(x) \) is chosen.

To prove this result first consider a differential operator of the form

\[ L_x u(x) = \frac{1}{w(x)} \frac{d}{dx} \left[ p(x) \frac{du(x)}{dx} \right] + c(x)u(x) \quad (7) \]

with \( w(x) > 0 \) and \( p(x) \) and \( c(x) \) real. We integrate by parts to compute the adjoint with weight \( w(x) \):

\[
\int_a^b w(x)dx v^*(x)L_x u(x) = \\
\int_a^b w(x)dx v^*(x) \left( \frac{1}{w(x)} \frac{d}{dx} \left[ p(x) \frac{du(x)}{dx} \right] + c(x)u(x) \right) = \\
\int_a^b dx \left( v^*(x) \frac{d}{dx} \left( p(x) \frac{du(x)}{dx} \right) + v^*(x)w(x)c(x)u(x) \right) = \\
\int_a^b dxu(x) \frac{d}{dx} \left( p(x) \frac{dv^*(x)}{dx} \right) + \int_a^b dxu(x) \frac{d}{dx} \left( p(x) \frac{dv^*(x)}{dx} \right) + v^*(x)w(x)c(x)u(x) + \\
\left[ v^*(x)p(x)u'(x) - v''p(x)u(x) \right]_a^b = \\
\int_a^b dxw(x)u(x) \left( \frac{1}{w(x)} \frac{d}{dx} \left( (p(x) \frac{dv^*(x)}{dx}) + v^*(x)c(x) \right) + \\
p(x)[v^*(x)u'(x) - v''u(x)]]_a^b. \]

This shows that \( L^* = L \). Comparing (7) with the general form

\[ L_x u(x) = a(x)u''(x) + b(x)u'(x) + c(x)u(x) \]

\[ L_x u(x) = \frac{p(x)}{w(x)}u'' + \frac{p'(x)}{w(x)}u'(x) + c(x) \]

shows that these will be identical if

\[ b(x)/a(x) = p'(x)/p(x); \quad p(x) = e^{\int_a^x dx' b(x')/a(x')} \]

and

\[ w(x) = \frac{p(x)}{a(x)} = \frac{e^{\int_a^x dx' b(x')/a(x')}}{a(x)} \]

This shows that any second order differential equation with real coefficients can be put in this form.

The boundary terms become

\[ p(x)[v^*(x)u'(x) - v''(x)u(x)]]_a^b = a(x)w(x)[v^*(x)u'(x) - v''(x)u(x)]_a^b \]

The following boundary condition are identical on the \( U \) and \( V \) subspaces
\( u(a) = u(b) = 0 \) (Dirichlet conditions)
\( u'(a) = u'(b) = 0 \) (Neumann conditions)

\( u(a) = u(b) \) and \( u'(a) = u'(b) \) (periodic conditions - here \( p(x) \) must also be periodic.

\( \alpha u(a) - u'(a) = \beta u(b) - u'(b) = 0 \) (general)

For each of these choices the adjoint boundary conditions have the same form and the vector spaces \( U \) and \( V \) are identical.

**Lecture 24 - Topic 2: Green’s functions**

Green’s functions provide another method for solving differential equations. The method is not automatic, because it is still necessary to compute the Green’s function.

The problem is to find solutions of the following two problems

\[ L_x u(x) = f(x) \]
\[ L_x^1 v(x) = h(x) \]

where \( u(x) \) satisfies homogeneous boundary conditions and \( v(x) \) satisfies the adjoint boundary conditions. These equations can be expressed as operator equations

\[ L_x |u\rangle = |f\rangle \]
\[ L_x^1 |v\rangle = |h\rangle \]

A Green’s operator, \( G \), if it exists, is a solution to the equation

\[ L_x G = I \]

where \( I \) is the identity. The adjoint Green’s operator, \( g \), if it exists, is a solution to the equation

\[ L_x^1 g = I \]

Recall the following identities on Hilbert spaces with weight functions

\[ \langle g| f \rangle = \int_a^b w(x)g^*(x)f(x)dx \]

which implies the representation of the identity

\[ I = \int |x\rangle w(x)dx \langle x| \]

Using \( I^2 = I \) with the representation above gives:

\[ \langle x|y \rangle = \frac{\delta(x-y)}{w(x)} \]
The inhomogeneous differential equation has the form
\[ L_x u(x) = f(x). \]

To find \( u(x) \) multiply this equation by \( g^*(x, y) \), integrate over \( x \) to get
\[ \int (g^*(x, y)) L_x u(x) w(x) dx = \int g^*(x, y) f(x) w(x) dx. \]

Using Green’s theorem this becomes
\[ \int g^*(x, y) f(x) w(x) dx = \int g^*(x, y) L_x u(x) w(x) dx = \int (L_x g(x, y))^* u(x) w(x) dx = \int \frac{\delta(x - y)}{w(x)} u(x) w(x) dx = u(y) \]
or
\[ u(y) = \int g^*(x, y) f(x) w(x) dx. \]

Similarly for \( L_x^* v(x) = f(x) \).

To find \( v(x) \) multiply both sides by \( G^*(x, y) \) and integrate over \( x \):
\[ \int (G^*(x, y)) L_x^* v(x) w(x) dx = \int G^*(x, y) f(x) w(x) dx. \]

Using Green’s theorem this becomes
\[ \int G^*(x, y) f(x) w(x) dx = \int (L_x G(x, y))^* v(x) w(x) dx = \int \frac{\delta(y - x)}{w(x)} v(x) w(x) dx = v(y) \]
or
\[ v(y) = \int G^*(x, y) f(x) w(x) dx. \]

where in these expressions as function of \( x \), \( G(x, y) \) satisfies boundary conditions while \( g^*(x, y) \) satisfies the adjoint boundary conditions.

**Lecture 24 - Topic 3: Properties of Green’s functions**

Using Green’s theorem with
\[ u(x) = G(x, z) \quad v(x) = g^*(x, y) \]
\[ \int g^*(x, y) L_x G(x, z) w(x) dx = \int (L_x g^*(x, y)) G(x, z) w(x) dx. \]
This gives the following relation between \( G(x, y) \) and \( g(x, y) \)

\[
g^*(z, y) = G(y, z)
\]

It follows that

\[
u(y) = \int g^*(x, y) f(x) w(x) dx = \int G(y, x) f(x) w(x) dx
\]

and

\[
v(y) = \int G^*(x, y) f(x) w(x) dx = \int g(y, x) f(x) w(x) dx
\]

These equations mean that we can use \( G(x, y) \) or \( g(x, y) \) to solve both

\[
L_x u(x) = f(x) \quad L^*_x v(x) = f(x)
\]

When \( L_x = L^*_x \) then

\[
G(x, y) = g(x, y) = G^*(y, x)
\]

which looks like a formal hermitian matrix; i.e. it is its complex conjugate transpose.

**Lecture 24 - Topic 4: Calculating Green’s functions**

Green functions are only useful if they can be computed. Since any second order differential operator with real coefficients is a self-adjoint operator on a space Hilbert space with weight \( w(x) \), in that case the Green’s function is a solution to

\[
L_x G(x, y) = \frac{\delta(x - y)}{w(x)} \quad (8)
\]

When \( x \neq y \) the right-side of this equation is zero and \( G(x, y) \), as a function of \( x \), is a solution of the homogeneous differential equation.

\[
(a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} + c(x)) G(x, y) = 0 \quad x \neq y.
\]

Using

\[
\frac{1}{p(x)} \frac{dp(x)}{dx} = \frac{b(x)}{a(x)} \quad p(x) = e^{\int_{x_0}^{x} \frac{b(x')}{a(x')} dx'}
\]

this equation becomes, including the delta-function term:

\[
\frac{d}{dx} \left( p(x) \frac{dG(x, y)}{dx} \right) + \frac{p(x) c(x)}{a(x)} G(x, y) = \frac{\delta(x - y) p(x)}{a(x) w(x)}.
\]

Integrating both sides of this equation over a small volume \((y - \epsilon, y + \epsilon)\) gives

\[
(p(y^+) \frac{dG(y^+, y)}{dx}) - (p(y^-) \frac{dG(y^-, y)}{dx}) + \int_{y-\epsilon}^{y+\epsilon} \frac{c(x) p(x)}{a(x)} G(x, y) dx = \frac{p(y)}{a(y) w(y)}
\]
Taking the limit $\epsilon \to 0$ gives

$$\frac{dG(y^+, y)}{dx} - \frac{dG(y^-, y)}{dx} = \frac{1}{a(y)w(y)}$$ (9)

If $G(x,y)$ is bounded near $x = y$ then the derivative of $G(x,y)$ with respect to $x$ has a discontinuity at $x = y$. Since the discontinuity is finite, integration over the discontinuity is continuous which implies that $G(x,y)$ is continuous at $x = y$.

The solution (8) involves using linear combinations of the independent solutions of the homogeneous equations that satisfy the boundary conditions at $a$ and $b$ and have the discontinuity (9) at $x = y$. In general there will be two solutions for $a \leq x \leq y$ and two more for $y \leq x \leq b$.

$$G(x, y) = \begin{cases} 
\alpha(y)u_1(x) + \beta(y)u_2(x) & x < y \\
\gamma(y)u_1(x) + \delta(y)u_2(x) & x > y
\end{cases}$$

where

$$\alpha(y)u_1(y) + \beta(y)u_2(y) = \gamma(y)u_1(y) + \delta(y)u_2(y)$$

$$\gamma(y)u_1'(y) + \delta(y)u_2'(y) - \alpha(y)u_1'(y) - \beta(y)u_2'(y) = \frac{1}{a(y)w(y)}$$

and $G(x, y)$ and $\frac{dG(x,y)}{dx}$ satisfy the same boundary conditions as $u(x), u'(x)$ at $x = a$ and $b$.

This assumes that $L_x$ has an inverse. This will happen if the boundary conditions uniquely fix all of coefficients. This can fail if there are non-zero solutions to $L_xu(x) = 0$ that satisfy both boundary conditions. Then it is necessary to use a more general construction.

29:4762 Lecture 25 - Friday - March 27

In this lecture I discuss

- Self-adjoint operators
- Green’s functions

Lecture 25 - Topic 1: Existence and uniqueness

Using the notation

$$G(x, y) = \begin{cases} 
\alpha(y)u_1(x) + \beta(y)u_2(x) & x < y \\
\gamma(y)u_1(x) + \delta(y)u_2(x) & x > y
\end{cases}$$

the boundary conditions at $x = y$ can be expressed as

$$\begin{pmatrix}
  u_1(y) & u_2(y) \\
  u_1'(y) & u_2'(y)
\end{pmatrix} \begin{pmatrix}
  \gamma(y) - \alpha(y) \\
  \delta(y) - \beta(y)
\end{pmatrix} = \begin{pmatrix}
  0 \\
  1/(a(y)w(y))
\end{pmatrix}$$
The determinant of the matrix is the Wronskian, which is not singular for independent solutions. This means that it is always possible to solve for the differences $\gamma(y) - \alpha(y)$ and $\delta(y) - \beta(y)$.

These relations can be used to investigate the boundary conditions at the endpoints. Define

$$
\eta(y) := \alpha(y) - \gamma(y) \quad \chi(y) := \beta(y) - \delta(y)
$$

$$
G(x, y) = \gamma(y)u_1(x) + \delta(y)u_2(x) + \begin{cases} 
\eta(y)u_1(x) + \xi(y)u_2(x) & x < y \\
0 & x > y
\end{cases}
$$

In this expression the quantities fixed by the boundary conditions at $x = y$ are in the expression for $G_-(x, y)$, while the unknown $\delta(y)$ and $\gamma(y)$ multiply the independent solutions.

Let $B_1(G)$ and $B_2(G)$ represent that boundary conditions satisfied by $G$ at $a$ and/or $b$. Because these are homogeneous boundary conditions they can be put in the form

$$
B_1(G) = B_1(G_-) + \gamma(y)B_1(u_1) + \delta(y)B_1(u_2) = 0
$$

$$
B_2(G) = B_2(G_-) + \gamma(y)B_2(u_1) + \delta(y)B_2(u_2) = 0
$$

The condition for a solution $\gamma(y)$, $\delta(y)$

$$
\det \begin{pmatrix} B_1(u_1) & B_1(u_2) \\ B_2(u_1) & B_2(u_2) \end{pmatrix} \neq 0
$$

If this determinant is 0 then there are non-zero constants $\gamma$ and $\delta$

$$
B_1(\gamma u_1 + \delta u_2) = 0 \quad B_2(\gamma u_1 + \delta u_2) = 0
$$

which means that there is a non-trivial solution of $L_x u = 0$ satisfying both boundary conditions.

When this does not happen the matrix can be inverted to solve for $\gamma$ and $\delta$. This ensures that the Green’s function exists and is unique. This leads to

$$
uf(x) = \int_a^b G(x, y)w(y) f(y)dy
$$

Lecture 25 - Topic 2: example 1

It is useful to illustrate this construction with some examples. The first one is trivial:

$$
L_x = \frac{d^2}{dx^2} \quad u(0) = u(a) = 0
$$

In this case we take $w(x) = 1$. The Green’s functions satisfies

$$
\frac{d^2G(x, y)}{dx^2} = \delta(x - y)
$$
In this case the independent solutions are

$$u_1(x) = 1 \quad u_2(x) = x$$

$$G(x, y) = \begin{cases} \alpha(y) + \beta(y)x & 0 < x < y \\ \gamma(y) + \delta(y)x & y < x < a \end{cases}$$

Boundary conditions at 0 give $\alpha(y) = 0$. The boundary condition at $x = a$ give $\gamma(y) = -a\delta(y)$. Continuity at $y$ gives

$$y\beta(y) = \delta(y)(y - a)$$

while the discontinuity of $G(x, y)$ at $y$ gives

$$\delta(y) - \beta(y) = 1$$

Solving gives

$$\beta(y) = (y - a)/a \quad \delta(y) = \frac{y}{a}$$

so the Green’s function for this operator (including boundary conditions) is

$$G(x, y) = \begin{cases} (y - a)x/a & 0 < x < y \\ (x - a)y/a & y < x < a \end{cases}$$

If we take $f(x) = 1$

$$u(x) = \int_0^a G(x, y)dy =$$

$$\int_0^x G(x, y)dy + \int_x^a G(x, y)dy =$$

$$\int_0^x (x - a)y/ady + \int_x^a (y - a)x/ady =$$

$$\frac{(x - a)x^2}{2a} - \left(\frac{x^2}{2} - ax\right)x/a + \left(\frac{a^2}{2} - a^2\right)x/a =$$

$$x^2/2 - ax/2$$

This satisfies the inhomogeneous equation and the boundary conditions. Note that the only linear combination of the form $u(x) = ax + b$ satisfying $u(0) = u(a) = 0$ is the trivial zero solution.

**Lecture 25 - Topic 3: example 2**

In this example we convert a differential operator $L_x$ with a weight $w(x)$ into an equivalent problem with $w(x) = 1$. To do this replace the differential operator $L_x$ with $w(x)L_x$. This product is self adjoint with respect to a norm with weight 1 provided $L_x$ is self adjoint with respect to weight $w(x)$. 

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We start with the original differential operator

\[ L_x = \frac{1}{w(x)} \frac{d}{dx} \left( p(x) \frac{d}{dx} \right) + c(x). \]

In this case the inhomogeneous differential equation

\[ w(x)L_x = f(x) \]

has the form

\[ \frac{d}{dx} \left( p(x) \frac{du(x)}{dx} \right) + w(x)c(x)u(x) = f(x) \quad a \leq x \leq b \]

with homogeneous boundary conditions

\[ u(a) = \alpha \frac{du(a)}{dx}, \quad u(b) = \beta \frac{du(b)}{dx}. \]

We let \( u_1(x) \) and \( u_2(x) \) be independent solutions of the homogeneous version of the differential equation. The Green’s function has the form

\[ G(x, y) = \begin{cases} 
  a_{11}(y)u_1(x) + a_{12}(y)u_2(x) & a \leq x < y \\
  a_{21}(y)u_1(x) + a_{22}(y)u_2(x) & y < x \leq b 
\end{cases} \]

First we apply the boundary conditions at \( a \) and \( b \):

\[ a_{11}(y)u_1(a) + a_{12}(y)u_2(a) = \alpha \left( a_{11}(y)u'_1(a) + a_{12}(y)u'_2(a) \right) \]
\[ a_{21}(y)u_1(b) + a_{22}(y)u_2(b) = \beta \left( a_{21}(y)u'_1(b) + a_{22}(y)u'_2(b) \right) \]

The solution in each region has the form

\[ U_<(x) := u_1(x) + \frac{a_{12}}{a_{11}} u_2(x) = \]
\[ u_1(x) - \frac{u_1(a) - \alpha u'_1(a)}{u_2(a) - \alpha u'_2(a)} u_2(x) \]
\[ U_>(x) := u_1(x) + \frac{a_{22}}{a_{21}} u_2(x) = \]
\[ u_1(x) - \frac{u_1(b) - \beta u'_1(b)}{u_2(b) - \beta u'_2(b)} u_2(x) \]

These also satisfy the boundary condition in each region.

Continuity at \( x = y \) gives

\[ a_{11}(y)U_<(y) = a_{21}(y)U_>(y) \]

while the discontinuity at \( y \) gives

\[ a_{21}(y)U'_>(y) - a_{11}(y)U'_<(y) = \frac{1}{p(y)} \]
(recall that in this case \(a(x)w(x) = p(x)\))

These equations can be solved to find the coefficients

\[
a_{11}(y) = \frac{U>(y)}{p(y)(U>_>(y)U<_<(y) - U>(y)U'_<(y))}
\]

\[
a_{21}(y) = \frac{U<_<(y)}{p(y)(U>_>(y)U<_<(y) - U>(y)U'_<(y))}
\]

This gives

\[
G(x, y) = \begin{cases} 
\frac{U>(y)U<_<(x)}{p(y)(U>_>(y)U<_<(y) - U>(y)U'_<(y))} & a \leq x < y \\
\frac{U<_<(y)U_>>(x)}{p(y)(U>_>(y)U<_<(y) - U>(y)U'_<(y))} & y < x \leq b
\end{cases}
\]

The denominator is the Wronskian of the solutions \(U_>(x)\) and \(U_<(x)\) of the homogeneous equation.

From the differential equation

\[
\frac{d}{dx}(p(x)\frac{d}{dx}(U_<>(x)U_>(x))) = \frac{d}{dx}(p(x)(U'_<(x)U_>(x) - U_<(x)U'_>(x))) = \]

\[
U_>(x)\frac{d}{dx}(p(x)U'_<(x)) - U_<(x)\frac{d}{dx}(p(x)U'_>(x)) = -(U_<(x)U_>(x))(w(x)c(p(x) - w(x)c(p(x)) + f(x) - f(x) = 0
\]

This means that

\[
p(x)(U'_<(x)U_>(x) - U_<(x)U'_>(x))
\]

is a constant. If \(U_<(x)\) and \(U_>(x)\) are independent solutions then their Wronskian is non zero and the constant must be non-zero.

Conversely if the constant is zero then

\[
U_<(x) = \eta U_>(x)
\]

Because these functions are proportional they satisfy the same homogeneous boundary conditions.

**Lecture 25 - Topic 4 : example 3**

In this example the Green’s function is used to convert the differential equation to an integral equation.

In this case consider the differential equation

\[
\frac{d}{dx}(x \frac{du(x)}{dx}) - \frac{u(x)}{x} = -\nu^2 xu(x)
\]
In this example we act as if the right side of the equation is an inhomogeneous term. Independent solutions of
\[
\frac{d}{dx}(x \frac{du}{dx}) - \frac{u(x)}{x} = 0
\]
are
\[
u_<(x) = x \quad u_>(x) = \frac{1}{x} - x.
\]
If we consider the boundary conditions \(u(0) = u(1) = 0\) we note that \(u_1(0) = 0\) and \(u_2(1) = 0\). Following example (2), \(p(x) = x\) and
\[
u_<(x)u'_>(x) - u'_<(x)u_>(x) = x\left(-\frac{1}{x^2} - 1\right) - \left(\frac{1}{x} - x\right) = -\frac{2}{x}
\]
The Green’s function has the form
\[
G(x, y) = \begin{cases} 
\frac{u_>(y)u_<(x)}{p(y)(u_>(y)u_<(y)) - u_>(y)u'_<(y))} & 0 \leq x < y \\
\frac{u_<(y)u_>(x)}{p(y)(u_<(y)u_>(y)) - u_<(y)u'_>(y))} & y < x \leq 1 \\
-\frac{1}{2y}(1 - y^2) & 0 \leq x < y \\
-\frac{y}{2x}(1 - x^2) & y < x \leq 1
\end{cases}
\]
It follows that
\[
u(x) = -\nu^2 \int_0^2 G(x, y)u(y)dy
\]
which has been used to transform the differential equation into an equivalent integral equation.

**Lecture 26 - Topic 1: Existence and uniqueness**

In this section we shown that the non-existence of non-trivial solutions of the homogeneous equation is a necessary and sufficient condition of the existence and uniqueness of the Green’s function.

We have already constructed a unique \(G(x, y)\) assuming there are no non-trivial solutions of the homogeneous equation
\[
L_x u(x) = 0
\]
obeying the boundary conditions. This shows that the absence of non-trivial solutions of the homogeneous equation satisfying the boundary conditions is a sufficient condition for the existence of a unique Green’s function.

Conversely assume that the Green’s function \(G(x, y)\) exists. It follows that
\[
\int_a^b w(x)(L^t v(x)) G(x, y)dx = \int_a^b w(x)v^*(x)(L_x G(x, y))dx = \\
\int_a^b w(x)v^*\left(\frac{1}{w(x)}\delta(x - y)\right) = v^*(x).
\]
If \((L^\dagger v(x))^* = 0\) then this equation requires that \(v(x) = 0\). This means that there are no non-trivial solutions of the adjoint equation satisfying the adjoint boundary conditions.

Since the adjoint equation has no non-trivial solutions, then there is a unique-adjoint Green function:

\[ L^\dagger g = I. \]

Since

\[ G(x, y) = g^*(y, x) \]

\(G(x, y)\) is also unique. This means that

\[ L_x u(x) = 0 \]

has no non-trivial solutions, because otherwise it could be added to \(G(x, y)\) which would make \(G(x, y)\) non-unique.

This shows that the absence of non-zero solutions of \(L_x u(x) = 0\) satisfying the boundary conditions is a necessary and sufficient condition for the existence of a unique Green’s function.

**Lecture 26 - Topic 2: Generalized Green’s functions**

For matrices the linear system

\[ M|u\rangle = |f\rangle \quad (10) \]

can have solutions for some choices of \(|f\rangle\) even when \(M\) does not have an inverse. If \(\langle v|\) is orthogonal to the range of \(M\) then

\[ \langle v|M|u\rangle = 0 \]

for any \(|u\rangle\). If (10) is to hold then \(|f\rangle\) must be in the range of \(M\) which requires

\[ \langle v|f\rangle = 0 \]

for any dual vector \(\langle v|\) orthogonal to the range of \(M\). If \(M\) is not invertible it will also have a null space of vectors satisfying \(M|w\rangle = 0\). Clearly if \(|u\rangle\) is a solution of the above equation \(|u\rangle + |w\rangle\) will be another solution. To get a unique solution we can require that it is orthogonal to the null space of \(M\). This is must be unique, otherwise the difference of 2 independent solutions would be in the null space of \(M\).

Thus we see the for matrices the above equation can be solved uniquely provided \(|f\rangle\) is in the range of \(M\) and \(|u\rangle\) is in the complement of the null space of \(M\).

In the matrix case equation (10) can be solved using the Moore-Penrose generalized inverse. Below I discuss the corresponding construction using Green’s functions. The object that replaces the Green’s function is called a generalized Green’s function.
First consider the conditions on $|f\rangle$ that are necessary for a solution. To do this consider the inhomogeneous equation

$$ L_x|u\rangle = |f\rangle. $$

Multiplying this equation on the left by a $\langle v|$ satisfying the adjoint boundary condition gives

$$ \langle v|f\rangle = \langle v|L_x|u\rangle = \langle (L_x^\dagger v)|u\rangle = 0. $$

This means that $\langle v|f\rangle = 0$ is a necessary condition for $L|u\rangle = |f\rangle$ to have a solution.

Similarly, using the same method, $\langle u|f\rangle = 0$ is a necessary condition for the adjoint $L^\dagger_x|v\rangle = |f\rangle$ to have a solution.

We know the cause of this difficulty is the existence of non-trivial solutions of the homogeneous equations satisfying the boundary conditions. These vectors are in the null space of $L_x$.

A generalized Green function is a solution of $L_x\tilde{G} = P$ where $P$ is a projection of the orthogonal complement of the null space of $L_x^\dagger$. This can be expressed as

$$ L_x\tilde{G}(x,y) = \frac{\delta(x-y)}{w(x)} - \sum_i \langle x|v_i\rangle\langle v_i|y\rangle $$

and for the adjoint equation

$$ L_x^\dagger\tilde{g}(x,y) = \frac{\delta(x-y)}{w(x)} - \sum_i \langle x|u_i\rangle\langle u_i|y\rangle. $$

These equations still do not necessarily imply unique solutions for $\tilde{G}(x,y)$ and $\tilde{g}(x,y)$. They should also satisfy

$$ \langle u|\tilde{G} = 0 \quad \langle v|\tilde{g} = 0 $$

where $|u\rangle$, and $|v\rangle$ are solutions of the homogeneous equations satisfying boundary conditions (resp. adjoint boundary conditions). The resulting solutions are unique, since otherwise we could add solutions to the homogeneous equation, resulting in non-unique solutions for $\tilde{G}(x,y)$ and $\tilde{g}(x,y)$.

This is most simply illustrated with an example. Let

$$ L_x = \frac{d^2}{dx^2} $$

and consider

$$ L_x u(x) = f(x) \quad -a \leq x \leq a $$

with periodic boundary conditions

$$ u(-a) = u(a); \quad u'(-a) = u'(a) $$

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The solution $u(x) = c$ is a non-trivial solution of this equation satisfying the periodic boundary conditions. We normalize this solution to unity

$$u(x) = \frac{1}{\sqrt{2a}}.$$ 

It is easy to check using integration by parts that $L = L'$ and the boundary conditions and adjoint boundary conditions are identical. The generalized Green’s function is a solution to

$$\frac{d^2}{dx^2} \tilde{G}(x, y) = \delta(x - y) - \frac{1}{2a}$$

In this case we look for solutions in the regions where the delta function vanishes

$$\tilde{G}(x, y) = \begin{cases} \alpha_+ + \beta_- x - \frac{x^2}{4a} & x < y \\ \alpha_+ + \beta_- x - \frac{x^2}{4a} & x > y \end{cases}$$

There are four quantities that have to be fixed to get a unique solution. Continuity at $x = y$ requires

$$\alpha_+ - \alpha_- + (\beta_+ - \beta_-) y = 0$$

while the discontinuity of the derivative at $x = y$ gives

$$\beta_+ - \beta_- = 1.$$ 

These equations imply

$$\alpha_+ - \alpha_- + y = 0 \quad \beta_+ - \beta_- = 1.$$ 

The periodic boundary conditions give

$$\alpha_- - \beta_- a - \frac{a}{4} = \alpha_+ + \beta_+ a - \frac{a}{4}$$

or

$$-a(\beta_+ + \beta_-) = \alpha_+ - \alpha_- = -y$$

and

$$\beta_- + \frac{1}{2} = \beta_+ - \frac{1}{2}$$

The discontinuity of the derivative at $x = y$ gives the same boundary condition as the periodicity of the derivatives (compare the boxed equations). Solving gives

$$\beta_+ = \frac{a + y}{2a} \quad \beta_- = \frac{a - y}{2a} \quad \alpha_+ = \alpha_- - y$$

which gives

$$\tilde{G}(x, y) = \begin{cases} -\frac{x^2}{4a} + \alpha_- + \frac{y-a}{2a} x & x < y \\ -\frac{x^2}{4a} + (\alpha_- - y) + \frac{y+a}{2a} x & y < x \end{cases}$$

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This is not uniquely determined since \( \alpha < \) is not determined. Requiring that
\[
\int_{-a}^{a} \frac{1}{\sqrt{2a}} \tilde{G}(x, y) \, dx = 0
\]
fixes \( \alpha < \) and gives the unique solution:
\[
\tilde{G}(x, y) = -\frac{x^2}{4a} + \frac{xy}{2a} - \frac{y^2}{4a} - \frac{a}{6} + \frac{1}{2}|x - y|
\]
You should check that this solution has all of the required properties.

**Lecture 26 - Topic 3: Inhomogeneous boundary conditions**

The class of boundary conditions that we have studied so far are homogeneous boundary conditions. They have the form
\[
B(u) := c_1 u(a) + c_2 u(b) + c_3 u'(a) + c_4 u'(b) = 0
\]
where the \( c_i \) are constants. It is also possible to impose boundary conditions of the form
\[
B(u) := c_1 u(a) + c_2 u(b) + c_3 u'(a) + c_4 u'(b) = d \neq 0.
\]
These are called inhomogeneous boundary conditions. The difference is that linear combinations of functions satisfying inhomogeneous boundary conditions do not satisfy the same boundary conditions. However, adding solutions that satisfy the homogeneous boundary conditions to a solution satisfying the inhomogeneous boundary conditions results in a class of functions satisfying the inhomogeneous boundary conditions.

To treat problems with inhomogeneous boundary conditions we let
\[
v(x) = g(x, y) = G(x, y)
\]
where \( G(x, y) \) the Green’s function associated with the homogeneous boundary conditions. We recall that if the coefficients of the differential equation are real the weight function can be chosen so \( L_x \) is Hermitian with respect to the weight. This means the Green’s function and it adjoint are the same.

We use this \( v(x) \) in the generalized Green’s identity where we do not assume that \( u(x) \) satisfies the homogeneous boundary conditions
\[
\int_{a}^{b} w(x)(L_x G(x, y)u(x) - G(x, y)L_x u(x)) \, dx =
\]
\[
\frac{1}{p(x)} \left( \frac{dG(x, y)}{dx} \right) u(x) - G(x, y)u'(x) \bigg|_{x=b}^{x=a}.
\]
We assume that \( L_x u(x) = f(x) \). Using
\[
L_x G(x, y) = \frac{1}{w(x)} \delta(x - y)
\]
and $L_x u(x) = f(x)$ in the generalized Green’s identity gives

$$u(y) = \int G(x, y) w(x) f(x) dx + \frac{1}{p(x)} \left( \frac{dG(x, y)}{dx} u(x) - G(x, y) u'(x) \right)_{x=0}^{x=b}.$$ 

So far we have not applied boundary conditions to $u(x)$. There are four unknowns - the homogeneous boundary conditions satisfied by $G(x, y)$ eliminate two of the unknowns. The inhomogeneous boundary conditions can be used in this expression to fix the remaining values of $u(x)$ and $u'(x)$ on the boundary.

An example is the best way to understand how this works. We look for a solution of

$$u''(x) = f(x)$$

on $[0, a]$ with inhomogeneous boundary conditions $u(0) = \sigma_1$ and $u(a) = \sigma_2$. The corresponding homogeneous boundary conditions are $u(0) = u(a) = 0$.

The independent solutions of the homogeneous equation are 1 and $x$. The Green function is a linear combination of 1 and $x$ satisfying the homogeneous boundary conditions, continuous at $x = y$ and has a derivative with a discontinuity of 1 at $x = y$. This was computed previously:

$$G(x, y) = \begin{cases} (y-a)x/a & x < y \\ (x-a)y/a & y < x \end{cases}.$$ 

Note that $g(x, y) = G(x, y)$ (This follows because $x \leftrightarrow y$, but the inequalities $x < y$ and $x > y$ also reverse). Inspection shows that $G(x, y)$ vanishes at $x = y$ and the derivative has a discontinuity of 1 at $x = y$. Note that

$$\frac{dG(x, y)}{dx} \begin{cases} (y-a)/a & x < y \\ y/a & x > y \end{cases}.$$ 

Solving the inhomogeneous equation for $f(x) = 1$ gives

$$u(y) = \int_0^y (y-a)x/adx + \int_y^a (x-a)y/adx + u(a)(y/a) - u(0)(y/a - 1) - (G(x, y)u'(x))|_{x=0}^{x=a}.$$ 

The Green’s function $G(x, y) = 0$ at $x = 0$ and $x = a$ which eliminates the coefficients of $u'(0)$ and $u'(a)$. This gives

$$u(y) = \int_0^y (y-a)x/adx + \int_y^a (x-a)y/adx(\sigma_2 - \sigma_1)(x/a) + \sigma_1 = y^2/2 - ay/2 + (\sigma_2 - \sigma_1)(y/a) + \sigma_1.$$ 

It is easy to check that this function has all of the required properties.

**Lecture 26 - Topic 4: Strum Liouville Problems**

In this section we study eigenvalue problems of the from

$$L_x |u_\lambda\rangle = \lambda |u_\lambda\rangle$$
where we assume that $L_x = L_x^\dagger$ and the solutions satisfy homogeneous boundary conditions associated with $L_x$. If we consider

$$L_x |u\rangle = |f\rangle$$

it has a solution

$$|u\rangle = G |f\rangle$$

which can be expressed as

$$\langle x | u \rangle = \int_a^b w(y) G(x, y) \langle y | f \rangle \, dy,$$

where existence assumes that there are no eigenfunctions of $L_x$ with eigenvalue $\lambda = 0$. We can use this write the eigenvalue equation as an integral equation

$$|u_\lambda\rangle = \lambda G |u_\lambda\rangle$$

which can be put in the form

$$G |u_\lambda\rangle = \frac{1}{\lambda} |u_\lambda\rangle.$$  

While $G(x, y)$ has discontinuous derivatives, it is a continuous function of $x$ and $y$ and the discontinuity at $x = y$ is finite. This means

$$\text{Tr}(G^\dagger G) = \int_a^b w(x) w(y) |G(x, y)|^2 < \infty.$$  

This means the $G(x, y)$ is the kernel of a Hilbert-Schmidt integral operator.

This means that $G(x, y)$ has a set of orthonormal eigenvectors with real eigenvalues (because $G = G^\dagger$) that accumulate to 0. In addition, because $G^{-1} = L_x$ exists, while the eigenvalues $\frac{1}{\lambda_n}$ get small, they never vanish. The $\lambda_n$ are the eigenvalues of $L_x$. They are all finite, but they accumulate at $\infty$.

**Lecture 27 - Topic 1: Series expansion of Green’s function**

Let $L_x$ be a Hermitian second order differential operator that has only non-zero eigenvalues. Consider the equation

$$(L_x - l) |u\rangle = |f\rangle \quad a \leq x \leq b.$$  

Assume the eigenvectors and eigenvalues of $L_x$ are known

$$L_x |u_n\rangle = \lambda_n |u_n\rangle$$

where the eigenvectors satisfy homogeneous boundary conditions.

Let $G_l(x, y)$ the Green’s function for $(L_x - l)$. It satisfies

$$(L_x - l) G_l(x, y) = \frac{\delta(x - y)}{w(x)}.$$  

(11)
We can expand \( G_l(x, y) \) in the complete set of eigenvectors of \( L_x \):

\[
G_l(x, y) = \sum_n \langle x | u_n \rangle c_n(l, y)
\]

where \( c_n(l, y) \) are expansion coefficients. \( G_l(x, y) \) satisfies the boundary condition as a function of \( x \) since each \( \langle x | u_n \rangle \) satisfies the boundary conditions. Because \( \langle x | u_n \rangle \) is an eigenstate of \( L_x \) it follows that

\[
(L_x - l)G_l(x, y) = (L_x - l) \sum_n \langle x | u_n \rangle c_n(y) = \sum_n (\lambda_n - l) \langle x | u_n \rangle c_n(l, y).
\]

Multiply by \( w(x) \langle u_m | x \rangle \) and integrate, using (11) and the orthonormality of the eigenfunctions to get

\[
\int w(x) \langle u_m | x \rangle (L_x - l)G_l(x, y) = \sum_n (\lambda_n - l) \int w(x) \langle u_m | x \rangle \langle x | u_n \rangle c_n(l, y)
\]

\[
\langle u_m | y \rangle = (\lambda_m - l) c_m(l, y).
\]

This gives the following expression for the expansion coefficients

\[
c_m(l, y) = \frac{\langle u_m | y \rangle}{\lambda_m - l}.
\]

Using these coefficients in the series expansion of the Green’s function gives

\[
G_l(x, y) = \sum_n \frac{\langle x | u_n \rangle \langle u_n | y \rangle}{\lambda_n - l}.
\]

This Green’s function exists for all values of \( l \) when \( l \) is not an eigenvalue of \( L_x \).

If \( l = \lambda_n \) then \( \langle x | u_n \rangle \) is a solution of the homogeneous equation

\[
(L_x - l)|u_n\rangle = 0
\]

in which case we know the Green’s function does not exist. Since all of the eigenvalues of \( L_x \) are real (because \( L_x = L^*_x \)), \( G_l(x, y) \) can be analytically continued to complex \( z = l \),

\[
G_z(x, y) = \sum_n \frac{\langle x | u_n \rangle \langle u_n | y \rangle}{\lambda_n - z}.
\]

If we integrate \( G_z(x, y) \) over a closed circle of radius \( R \) as \( R \to \infty \), containing all of the eigenvalues, then we pick up a residue from each eigenvalue. Using completeness of the eigenfunctions gives

\[
\oint dz G_z(x, y) = -2\pi i \sum_n \langle x | u_n \rangle \langle u_n | y \rangle = -2\pi i \delta(x - y)
\]

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These relations are best illustrated with an example. Consider the second order differential equation

\[ u''(x) + lu(x) = f(x) \]

with homogeneous boundary conditions

\[ u(0) = u(a) = 0. \]

The solutions of the homogeneous equation, \( Lx|u⟩ = \lambda|u⟩ \), are linear combinations of

\[ \sin(\sqrt{\sigma}x) \quad \text{and} \quad \cos(\sqrt{\sigma}x). \]

Note that in order to satisfy the boundary conditions \( \lambda = -\sigma \) must be negative. The boundary conditions determine \( \sigma \)

\[ u_n(x) = \sqrt{\frac{2}{a}} \sin(\frac{n\pi}{a}x) \]

where these have been normalized to unity. The eigenvalues are

\[ \lambda_n = -\frac{n^2\pi^2}{a^2} = -\sigma \]

The series representation of the Green’s function is

\[ G_z(x,y) = -\sum_{n=1}^{\infty} \frac{2}{a} \frac{\sin(\frac{n\pi}{a}x) \sin(\frac{n\pi}{a}y)}{\sqrt{\frac{n^2\pi^2}{a^2}}} \sin(\frac{n\pi}{a}x) \sin(\frac{n\pi}{a}y). \]

On the other hand a direct calculation of the Green’s function that vanishes at \( x = 0 \) and \( x = a \) has the form

\[ G_I(x,y) = \begin{cases} a_<(y) \sin(\sqrt{l}x) & x < y \\ a_>(y) \sin(\sqrt{l}(x-a)) & x > y \end{cases}. \]

Continuity at \( x = y \) gives

\[ a_<(y) \sin(\sqrt{l}y) = a_>(y) \sin(\sqrt{l}(y-a)) \]

while the discontinuity of the derivative gives

\[ \sqrt{l}a_>(y) \cos(\sqrt{l}(y-a)) - \sqrt{l}a_<(y) \cos(\sqrt{l}y) = 1. \]

Solving for the coefficients \( a_>(y) \) and \( a_<(y) \), using some trig identities, gives

\[ G_I(x,y) = \begin{cases} \frac{\sin(\sqrt{l}x) \sin(\sqrt{l}(a-y))}{\sqrt{l}(\sin(\sqrt{l}a))} & x < y \\ \frac{\sin(\sqrt{l}(a-x)) \sin(\sqrt{l}y)}{\sqrt{l}(\sin(\sqrt{l}a))} & y < x \end{cases}. \]
It is easy to check that this satisfies the boundary conditions at $x = 0$ and $x = a$, is continuous at $x = y$, and the derivative has a discontinuity of 1 at $x = y$. This has poles in $l$ when \[
\sin(\sqrt{l}a) = 0 \quad \sqrt{l}a = n\pi.\]
The residue at each pole can be calculated using more trig identities to get \[
\text{Res} = -\frac{2}{a} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)
\]
which agrees with the residue of the series expression for the Green’s function.

Lecture 27 - Topic 2: Series solution of the differential equation

One way to solve differential equations is to use power series methods. This is called the “Frobenius method”. The advantage of this method is that for appropriate conditions on the coefficient functions the solutions are analytic functions, except possibly for some isolated singularities.

For this method consider a homogeneous differential equation of the form
\[
\frac{d^2 u(z)}{dz^2} + p(z) \frac{du(z)}{dz} + q(z) u(z) = 0
\]
where $z$ is a complex variable and $p(z)$ and $q(z)$ are analytic, except at some isolated singularities, in a region $R$.

The character of the solution in the neighborhood of a point $z \in R$ depends on some properties of the differential equation. Points in $R$ can are classified as follows:

- $z \in R$ is an ordinary point if $p(z)$ and $q(z)$ are analytic at $z$.
- $z \in R$ is a regular singular point if $p(z)$ has a pole of order $\leq 1$ and $q(z)$ has a pole of order $\leq 2$.
- $z \in R$ is irregular singular point if $p(z)$ and $q(z)$ are singular at $z$ and $z$ is not a regular singular point.

In what follows we will show that if $z_0$ is an ordinary point, the solution satisfying the boundary conditions is analytic in a neighborhood of $z_0$. If $z_0$ is a regular singular point the two solutions of (12) have the form
\[
\begin{align*}
u_1(z) &= (z - z_0)^{r_1} \sum_{n=0}^{\infty} c_n^1 (z - z_0)^n \\
u_2(z) &= (z - z_0)^{r_2} \sum_{n=0}^{\infty} c_n^2 (z - z_0)^n
\end{align*}
\]
with $r_1 \neq r_2$ or
\[
\begin{align*}
u_1(z) &= (z - z_0)^{r_1} \sum_{n=0}^{\infty} c_n^1 (z - z_0)^n \\
u_2(z) &= (z - z_0)^{r_2} \sum_{n=0}^{\infty} c_n^2 (z - z_0)^n
\end{align*}
\]
$u_2(z) = (z - z_0)^2 \sum_{n=0}^{\infty} c_n^2(z - z_0)^n + c_1 u_1(z) \ln(z - z_0)$

While we will not discuss the case of irregular singular points, it can be shown that the solutions in a neighborhood of an irregular singular point have the same form, but the series may extend from $n = -\infty$ to $n = \infty$.

Lecture 27 - Topic 3: Convergence near ordinary points

We assume that $z_0$ is an ordinary point. We seek a solution of

$$\frac{d^2 u(z)}{dz^2} + p(z) \frac{du(z)}{dz} + q(z) u(z) = 0$$

in a neighborhood of $z_0$ satisfying the boundary conditions

$$u(z_0) = a \quad \frac{du}{dz}(z_0) = b.$$

To solve this problem we use a transformation to eliminate the first derivative term. To do this express the solution in the form

$$u(z) = f(z) e^{-\frac{1}{2} \int_{z_0}^{z} p(z')dz'}$$

where the integral between $z_0$ and $z$ is along a path in the domain of analyticity of $p(z)$. In this case the integral is an analytic function of $z$ and the exponential of an analytic function is analytic, so the function multiplying $f(z)$ is analytic. Using this form of $u(z)$ in the differential equation gives the following expressions for $u''(z)$ and $u'(z)$:

$$u'(z) = (f'(z) - \frac{1}{2} f(z)p(z)) e^{-\frac{1}{2} \int_{z_0}^{z} p(z')dz'}$$

$$u''(z) = (f''(z) - \frac{1}{2} (f'(z)p(z) + p'(z)f(z)) - \frac{1}{2} p(z)f'(z) + \frac{1}{4} f(z)p^2(z)) e^{-\frac{1}{2} \int_{z_0}^{z} p(z')dz'} = (f''(z) - f'(z)p(z) - \frac{1}{2} p'(z)f(z) + \frac{1}{4} f(z)p^2(z)) e^{-\frac{1}{2} \int_{z_0}^{z} p(z')dz'}.$$

When these expressions are used in the differential equations it becomes

$$u''(z) + p(z)u'(z) + q(z)u(z) = (f''(z) - \frac{1}{2} p'(z)f(z) - \frac{1}{4} f(z)p^2(z) + f(z)q(z)) e^{-\frac{1}{2} \int_{z_0}^{z} p(z')dz'} = 0$$

where we can see that there are no $f'(z)$ terms. Since $e^{-\frac{1}{2} \int_{z_0}^{z} p(z')dz'}$ is never 0, the condition for $u(z)$ to satisfy the differential equation is that $f(z)$ is a solution of

$$f''(z) + k(z)f(z) = 0 \quad k(z) := q(z) - \frac{1}{4} p^2(z) - \frac{1}{2} p'(z).$$

(13)
We show that the following sum converges uniformly to an analytic solution of (13)

\[ f(z) = \sum_{n=0}^{\infty} f_n(z) \]

where the \( f_n(z) \) are defined by

\[ \frac{d^2 f_n(z)}{dz^2} = -k(z)f_{n-1}(z) \]

and

\[ f_0(z) = a' + b'(z - z_0) \]

is chosen so \( u(z) \) satisfies the boundary conditions.

Integrating the expression for \( f_n(z) \) twice gives

\[ f_n(z) = -\int_{z_0}^{z} dz' \int_{z_0}^{z'} dz'' k(z'')f_{n-1}(z''). \]

This integral can be expressed as (using integration by parts) as:

\[ f_n(z) = -\int_{z_0}^{z} dz' \frac{d}{dz'} z' \int_{z_0}^{z'} dz'' k(z'')f_{n-1}(z''). \]

We define

\[ f_0 := \max |f_0(z)| \quad z \in R \]

and

\[ k := \max |k(z)| \quad z \in R. \]

These are finite since \( k(z) \) and \( f_0(z) \) are analytic in \( R \) (we can replace \( R \) by a bounded disk in \( R \) centered at \( z_0 \)). By these definitions

\[ |f_0(z)| \leq f_0 k_0 \left| \frac{z - z_0}{0!} \right|^2 = f_0. \]

Next we assume by mathematical induction that for \( m < n \) that

\[ |f_m(z)| \leq f_0 k_m \left| \frac{z - z_0}{m!} \right|^{2m}. \]

Since the integrals are contour integrals we choose to compute the integrals using a straight line path between \( z_0 \) and \( z \). This will be in \( R \) for sufficiently small \( |z - z_0| \):

\[ z'(t) = z_0 + (z - z_0)t \quad dz' = (z - z_0)dt \quad 0 \leq t \leq 1 \]
For this path (14) becomes

\[ f_n(z) = (z - z_0)^2 \int_0^1 (t - 1)k(z(t))f_{n-1}(z(t)). \]

It follows by the induction assumption (15) that

\[ |f_n(z)| \leq f_0 |(z - z_0)^2| k \int_0^1 \frac{(t - 1)^{2n-1}}{(n-1)!} k^{n-1} |z - z_0|^{2n-2} = f_0 |(z - z_0)^{2n}| \frac{k^n}{(n-1)!} \int_0^1 (t - 1)^{2n-1} dt = \frac{1}{2} f_0 |(z - z_0)^{2n}| \frac{k^n}{n!} < f_0 |(z - z_0)^{2n}| \frac{k^n}{n!} \]

which shows that if (15) holds for \( m < n \) it holds for \( n \). Note that the \( f_n(z) \) for \( n \neq 0 \) are proportional to \( (z - z_0)^2 \) so they vanish on the boundary, \( f_n(z_0) = f'_n(z_0) = 0 \).

We define the function

\[ f(z) = \sum_{n=0}^{\infty} f_n(z) \]

and

\[ u(z) = f(z)e^{-\frac{1}{2} \int_{z_0}^{z} p(z')dz'} \]

The bounds on \( f_n(z) \) imply

\[ \sum_{n=0}^{\infty} |f_n(z)| \leq \sum_{n=0}^{\infty} f_0 |(z - z_0)^{2n}| \frac{k^n}{n!} = f_0 e^{k|(z - z_0)^2|} < \infty. \]

This means that this sum converges uniformly for \( z \) is any circle centered at \( z_0 \) where \( k(z) \) is analytic. Since each term in the series is analytic and this is a uniformly convergent sum of analytic functions and it is analytic (see page 45 of the text equations 17.8 and 17.9). The uniform convergence means the derivatives commute with the infinite sum (17.9) which gives

\[ \frac{d^2}{dz^2} \sum_{n=0}^{\infty} f_n(z) = -k(z) \sum_{n=1}^{\infty} f_{n-1}(z) - k(z) \sum_{n=0}^{\infty} f_n(z) = -k(z)f(z) \]

where we used \( \frac{d^2}{dz^2} f_0(z) = 0 \).

This shows that this series is a solution to the differential equation (13). We note that \( e^{-\frac{1}{2} \int_{z_0}^{z} p(z')dz'} \) is also analytic, so \( u(z) \) is analytic since it is a product of analytic functions.
Since each \( f_n(z) \) for \( n > 0 \) has a coefficient of \((z - z_0)^2\) they do not contribute to the boundary conditions. The boundary conditions can be satisfied by choosing \( a' \) and \( b' \) as follows:

\[
\begin{align*}
  u(z_0) &= a = f_0(z_0) = a' \\
  u'(z_0) &= b = f'(z_0) - \frac{1}{2} f(z_0) p(z_0) = b' - \frac{1}{2} a p(z_0)
\end{align*}
\]

which gives

\[
\begin{align*}
  a' &= a \\
  b' &= b + \frac{1}{2} a p(z_0)
\end{align*}
\]

We note that the solution is unique because if

\[
w(z) = f_1(z) - f_2(z)
\]

then \( w(z) \) is also a solution of the same equation with

\[
w(z_0) = w'(z_0) = 0.
\]

Since the first two derivatives of \( w(z) \) vanish at \( z - z_0 \), by differentiating the differential equation it follows all higher derivatives vanish; for example

\[
\begin{align*}
  w''(z_0) &= -k(z_0) w(z_0) \\
  w'''(z_0) &= -k'(z_0) w(z_0) - k(z_0) w'(z_0) \\
  &\vdots
\end{align*}
\]

By Taylor’s theorem for analytic functions we must have \( w(z) = 0 \), so the solution discussed above is unique. What we have shown is that if \( p(z) \) and \( q(z) \) are analytic in a region containing \( z_0 \), then there is a unique solution to the differential equation that is analytic in a disk centered at \( z_0 \) in the region \( R \).

In general the solution can be analytically continued, so the function is analytic at points that in \( R \) that cannot be reached by a straight line, however if \( R \) is multiply connected the analytic continuation could be path dependent and the resulting \( u(z) \) could be multivalued.

Since the solution is analytic we know that the solution has a series representation

\[
u(z) = \sum_{n=0}^{\infty} u_n (z - z_0)^n.
\]

The coefficients can be determined by substituting this expression in the equations (also expanding \( p(z) \) and \( q(z) \) about \( z = z_0 \))

For example assume that \( z = 0 \) is an ordinary point and

\[
\begin{align*}
p(z) &= \sum_{n=0} p_n z^n \\
q(z) &= \sum_{n=0} q_n z^n
\end{align*}
\]

and we look for a solution

\[
u = \sum_{n=0} u_n z^n.
\]
Then using all three expansions in the differential equation gives

\[ 0 = \frac{d^2 u(z)}{dz^2} + p(z) \frac{du(z)}{dz} + q(z)u(z) = \]

\[ \sum_{n=2}^{\infty} n(n-1)u_n z^{n-2} + \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} m u_m p_k z^{m-1+k} + \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} u_m q_k z^{m+k}. \]

Shifting the sums to get coefficients with powers of \( z^n \) gives

\[ \sum_{n=0}^{\infty} (n+2)(n+1)u_{n+2} z^n + \sum_{n=0}^{\infty} \sum_{m=0}^{n} (m+1)u_{m+1} p_{n-m} z^n + \sum_{n=0}^{\infty} \sum_{m=0}^{n} u_k q_{n-k} z^n = 0. \]

Equating coefficients of \( z^n \) gives

\[ (n+2)(n+1)u_{n+2} + \sum_{m=0}^{n} (m+1)u_{m+1} p_{n-m} + \sum_{m=0}^{n} u_m q_{n-m} = 0. \]

For \( n = 0 \) we get

\[ u_2 = -\frac{u_1 p_0 + u_0 q_0}{2} \]

where \( u_0 = u(0) \) and \( u_1 = u'(0) \). The rest of the coefficients can be found using

\[ u_{n+2} = -\frac{\sum_{m=0}^{n} ((m+1)u_{m+1} p_{n-m} + u_m q_{n-m})}{(n+2)(n+1)} \]

where the term on the right involves the \( u_k \) for \( k < n + 2 \). By the theorem, if \( z = 0 \) is an ordinary point this series converges uniformly in a neighborhood of 0.

**Lecture 27 - Topic 4: Convergence near regular singular points**

Let \( z_0 \) be a regular singular point and consider the differential equation

\[ \frac{d^2 u(z)}{dz^2} + p(z) \frac{du(z)}{dz} + q(z)u(z) = 0. \]

For regular singular points

\[ (z - z_0)p(z) \]

and

\[ (z - z_0)^2 q(z) \]

are analytic at \( z = z_0 \). This means that these products can be expressed as uniformly convergent power series about \( z_0 \):

\[ (z - z_0)p(z) := A(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \]
and
\[(z - z_0)^2 q(z) := B(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n.\]

The coefficients \(a_n\) and \(b_n\) can be computed by contour integration around a small counterclockwise circle, \(C\), in the domain of analyticity centered at \(z_0\):
\[
a_n = \frac{1}{2\pi i} \oint_C \frac{(z' - z_0)p(z')dz'}{(z' - z_0)^{n+1}},
\]
\[
b_n = \frac{1}{2\pi i} \oint_C \frac{(z' - z_0)^2 q(z')dz'}{(z' - z_0)^{n+1}}.
\]

As in the case of ordinary points, we assume a power series solution, except we factor out an unknown power of \((z - z_0)\):
\[u(z) = (z - z_0)^r \sum_{n=0}^{\infty} u_n (z - z_0)^n\]
where \(r\) will be chosen to have non-trivial boundary conditions and \(u_n\) will be determined from the differential equation and boundary conditions. In general \(r\) will not be an integer and may even be complex.

As in the example at the end of the topic 3 we substitute the series into the differential equation and equate the coefficients of like powers of \((z - z_0)^{r+n}\) to 0:
\[
0 = \frac{d^2u(z)}{dz^2} + p(z) \frac{du(z)}{dz} + q(z)u(z) =
\]
\[
\sum_{n=0}^{\infty} (r+n)(r+n-1)u_n (z - z_0)^{r+n-2} + \sum_{n=0}^{\infty} (r+n)u_n (z - z_0)^{r+n-2} \sum_{m=0}^{\infty} a_m (z - z_0)^m +
\]
\[
\sum_{n=0}^{\infty} u_n (z - z_0)^{r+n-2} \sum_{m=0}^{\infty} b_m (z - z_0)^m.
\]

Shifting indices this becomes
\[
= \sum_{n=0}^{\infty} (r+n)(r+n-1)u_n (z - z_0)^{r+n-2} + \sum_{n=0}^{\infty} \sum_{m=0}^{n} (r+m)a_{n-m}u_m (z - z_0)^{r+n-2} +
\]
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{n} b_{n-m}u_m (z - z_0)^{r+n-2} = 0.
\]

Setting the coefficient of each \((z - z_0)^{r+n-2}\) to zero gives
\[
u_n[(r+n)(r+n-1)+(r+n)a_0+b_0]+\sum_{m=0}^{n-1}[(r+m)a_{n-m}+b_{n-m}]u_m = 0 \quad (16)
\]
which expresses \( u_n \) in terms of the \( u_k \) for \( k < n \). For \( n = 0 \) this equation becomes

\[
u_0[r^2 - r + ra_0 + b_0] = 0.
\]

If

\[
[r^2 - r + ra_0 + b_0] \neq 0
\]

then \( u_0 = 0 \) and it follows from (16) that all of the \( u_n \) are zero. Thus in order to get a non-zero solution it is necessary to choose \( r \) as one of the roots of the quadratic polynomial:

\[
P(r) = r^2 + (a_0 - 1)r + b_0 = 0.
\]

This is called the \textit{indicial equation}. It is a second degree polynomial so it has 2 roots

\[
r_{\pm} = \frac{1 - a_0}{2} \pm \sqrt{(\frac{1 - a_0}{2})^2 - b_0}.
\]

Note that

\[
r_{+} + r_{-} = \frac{1 - a_0}{2}.
\]

If these roots have real parts, it is useful to classify them by

\[
Re(r_1) \geq Re(r_2)
\]

where the form of the roots of the quadratic equation imply

\[
Re(r_1) \geq Re(\frac{1 - a_0}{2})
\]

if at least one of the roots has a real part. The text defines two functions:

\[
\lambda_0(r) = P(r) = r(r - 1) + ra_0 + b_0
\]

and

\[
\lambda_m(r) \begin{cases} ra_m + b_m & m > 0 \\ 0 & m < 0 \end{cases}
\]

Using this notation (16) has the form

\[
u_n\lambda_0(r + n) = - \sum_{m=0}^{n-1} \lambda_{n-m}(r + m)u_m
\]

(18)

where \( \lambda_0(r + n) \) and \( \lambda_{n-m}(m + 2) \) are functions (not products) of \( (r + n) \) and \( (r + m) \) respectively.

The roots above are solutions to \( \lambda_0(r_{\pm}) = 0 \). If \( r \) is a root satisfying

\[
\lambda_0(r) = P(r) = r(r - 1) + ra_0 + b_0 = 0
\]

and if

\[
\lambda_0(r + n) = n^2 + n(2r + a_0 - 1) = 0
\]

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then \( r + n \) is also a root which means \( r_+ - r_- = n \) is an integer. \( \lambda_0(r + n) \) is never zero for \( n > 0 \) if \( r_+ - r_- \) is not an integer. In addition, it is always non zero for \( r = r_+ \), since adding \( n \) increases the real part, and by assumption \( r_+ \) is the root with the largest real part.

Thus for the case \( r = r_+ \) the relation between successive coefficients, \( u_n \), is

\[
    u_n = -\sum_{m=0}^{n-1} \frac{\lambda_{n-m}(r_+ + m)}{\lambda_0(r_+ + n)} u_m. \tag{19}
\]

When \( r_+ - r_- \neq n \) or when one or both of the roots are pure imaginary there is a second solution for the coefficients

\[
    u_n = -\sum_{m=0}^{n-1} \frac{\lambda_{n-m}(r_- + m)u_m}{\lambda_0(r_- + n)}. \tag{20}
\]

When \( r_+ - r_- = n \) is a positive integer the denominator on the right side of the iterative equation for the second root vanishes for some \( n \), so we can’t find the coefficients \( u_n \) for the second solution.

In the case where there are two series solutions it is possible to show uniform convergence. While there could be a branch cut due to the multivalued nature of \( (z - z_0)^r \), the part of the solution that multiplies \( (z - z_0)^r \) can be shown to be uniformly convergent.

To show convergence, since \( A(z) \) and \( B(z) \) are analytic in a disk \( D \) centered at \( z_0 \), it follows that

\[
    |A(z)| < A_1 \quad |B(z)| < A_2
\]

and

\[
    |a_n| \leq \frac{A_1}{R^n} \quad |b_n| \leq \frac{A_2}{R^n}
\]

for all \( z \in D \), where \( R \) is the radius of \( D \). These bounds follow from the Cauchy integral formula (see equation 18.7 on page 46 of text). Using these bounds in

\[
    |\lambda_m(r + m)| \leq |(r + m)|a_{n-m} + b_{n-m}| \leq \frac{|(r + m)|\lambda_1 + \lambda_2}{R^{n-m}}
\]

gives

\[
    \left| \frac{|r + m|\lambda_1 + \lambda_2}{R^{n-m}} \right|.
\]

In addition

\[
    |\lambda_0(r + m)| = |n^2 + n(2r - 1 + a_0)| \geq n^2
\]

which follows from (17) provided \( r = r_+ \), since \( r_+ + r_- = (1 - a_0)/2 \).

Using these bounds in the recursion implies

\[
    |u_n| \leq \frac{1}{n^2} \sum_{m=0}^{n-1} \frac{|r + m|\lambda_1 + \lambda_2}{R^{n-m}} |u_m|.
\]

For \( n \geq 1 \):

\[
    \frac{|r + m|\lambda_1 + \lambda_2}{n} \leq \frac{m}{n} \lambda_1 + \lambda_2.
\]

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Using this in the above inequality gives

\[ |u_n| \leq \frac{1}{n} \sum_{m=0}^{n-1} \left( |r| + \frac{m}{n} \Lambda_1 + \Lambda_2 \right) \frac{|u_m|}{R^{n-m}} \]

\[ \leq K \frac{n}{n} \sum_{m=0}^{n-1} \frac{|u_m|}{R^{n-m}} \]

where

\[ K = |r| + \Lambda_1 + \Lambda_2. \]

If we consider the structure of these inequalities

\[ |u_1| \leq \frac{K}{R} |u_0| \]

\[ |u_2| \leq \frac{K}{2} \frac{|u_0|}{R^2} + |u_0| \frac{K}{R^2} \leq \frac{K^2}{R^2} |u_0| \]

\[ |u_3| \leq \frac{K}{3} \left( \frac{1}{R^3} + \frac{K}{R^3} + \frac{K^2}{R^3} \right) |u_0| \leq \frac{K^3}{R^3} |u_0| \]

\[ \vdots \]

where we have assumed \( K > 1 \) otherwise we can replace \( K \) by 1. This can be continued for any \( n \) to get bounds on the coefficients in the series solution

\[ |u_n| \leq \frac{K^n}{R^n} |u_0|. \]

These bounds show that that

\[ \sum_{n=0}^{\infty} \frac{K^n |(z - z_0)|^n}{R^n} |u_0| \]

converges uniformly for

\[ \frac{K |(z - z_0)|}{R} < 1. \]

When \( r_+ - r_- = N \) the series solution is only valid for the solution with the root \( r = r_+ \). To get the other solution we use the method discussed in the first topic lecture 23. In that case we showed that the independent second solution can be expressed as

\[ u_2(z) = u_1(z) \int_{z_0}^{z} \frac{dz''}{u_1''(z'')} e^{-\int_{z_0}^{z''} p(z') dz'} \]

\[ (21) \]

where

\[ p(z) = \frac{A(z)}{z - z_0} = \frac{a_0}{z - z_0} + \sum \frac{a_n}{(z - z_0)^{n-1}}. \]
\[- \int_{z_0}^{z} p(z') dz' = -a_0 \ln(z - z_0) - \sum_{n=1}^{\infty} \frac{a_n}{n} (z - z_0)^n \]

resulting in a second solution of the form

\[ u_2(z) = u_1(z) \int_{z_0}^{z} \frac{dz'}{u_1^2(z')} e^{-a_0 \ln(z' - z_0) - \sum_{n=1}^{\infty} \frac{a_n}{n} (z' - z_0)^n} \]

Note that (21) can be expressed in the form

\[ \frac{d}{dz} \left( \frac{u_2(z)}{u_1(z)} \right) = \frac{c}{u_1(z)} e^{-a_0 \ln(z' - z_0) - \sum_{n=1}^{\infty} \frac{a_n}{n} (z' - z_0)^n} \]

where the \( u_n \) are the expansion coefficients for \( u_1(z) \) and

\[ F(z) = e^{-\sum_{n=1}^{\infty} \frac{a_n}{n} (z - z_0)^n} \left( \sum_{n=1}^{\infty} u_n (z - z_0)^n \right)^2 \]

which is analytic at \( z = z_0 \). \( F(z) \) can be expanded in a Taylor series about \( z_0 \).

The indicial equation gives

\[ r_1 + r_2 = 1 - a_0 \]

since \( r_1 = N + r_2 \) it follows that

\[ 2r_1 + a_0 = 1 + N \]

which implies

\[ \frac{d}{dz} \left( \frac{u_2(z)}{u_1(z)} \right) = (z - z_0)^{-2r - a_0} F(z) = (z - z_0)^{-(N+1)} F(z) = \sum_{n=0}^{\infty} f_n (z - z_0)^{n - 1 - N}. \]

When this is integrated it becomes

\[ u_2(z) = u_1(x) \left( (f_N \ln(z - z_0) + (z - z_0)^{-N} \sum_{n=0, n \neq N}^{\infty} \frac{f_n}{n - N} (z - z_0)^n) \right). \]

The coefficients can also be determined by recursion by substituting this series in the differential equation.
Note typically at least one of the two solutions will be multivalued. This is because when the roots of the indicial equation differ by integers the solution has the above form with has a log, while if the roots do not differ by integers, one of the roots is not an integer, which means \((z - z_0)^r\) with a non-integer \(r\) will be multivalued.

Lecture 28 - Topic 1: Integral representations

An integral representation of a function \(u(z)\) has the form

\[
u(z) = \int_a^b K(z, t)v(t)dt\]

along some path. If \(u(z)\) is a solution of the differential equation satisfying boundary conditions in some region \(D\), and this solution agrees with a series solution defined in a region \(R\) where \(R \cap D\) includes the neighborhood of a point, then the solutions, both being analytic functions must agree. It often happens that using the integral representation extends the domain of analyticity. We found this with the Gamma function, where we had two integral representations, and one had a larger domain of analyticity.

Example 1: Consider the series

\[
\sum_{n=0}^{\infty} z^n.
\]

This converges to \(1/(1 - z)\) for \(|z| < 1\), where it is analytic in this region. Consider the integral

\[
f(z) := \int_0^\infty e^{zt}e^{-t}dt.
\]

This is analytic for \(\text{Re}(z) < 1\) which includes large negative values of the real part of \(z\). Expanding the exponent

\[
f(z) = \int_0^\infty e^{zt}e^{-t}dt = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_0^\infty t^n e^{-t}dt = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Gamma(n + 1) = \sum_{n=0}^{\infty} z^n.
\]

These two analytic functions agree for \(|z| < 1\), so this integral representation extends the domain of analyticity from \(|z| < 1\) to \(\text{Re}(z) < 1\).

Example 2: Consider the differential equation

\[
L_z u(z) = 0.
\]

Assume that we have found \(K(z, t)\) and a differential operator \(M_t\) in \(t\) satisfying

\[
L_z K(z, t) = M_t K(z, t).
\]

Then

\[
L_z u(z) = \int_a^b L_z K(z, t)v(t)dt = \int_a^b M_t K(z, t)v(t)dt.
\]
If \( M^\dagger_t \) is adjoint of \( M_t \) then
\[
v(t)M_tK(z,t) - K(z,t)M^\dagger_t v(t) = \frac{\partial}{\partial t}Q(K(z,t)v(t)).
\]
Integrating \( t \) from \( a \) to \( b \) gives
\[
L_z u(z) = \int_a^b L_z K(z,t)v(t)dt = \int_a^b K(z,t)M^\dagger_t v(t) + Q(K(z,t)v(t)|_{t=a}^{t=b}).
\]
This will vanish provided \( Q(K(z,t)v(t)) \) vanishes at the endpoints and \( v(t) \) is a solution of the adjoint equation
\[
M^\dagger_t v(t) = 0.
\]
It follows that
\[
u(z) = \int_a^b K(z,t)v(t)dt
\]
is a solution to the equation since
\[
L_z u(z) = \int_a^b L_z K(z,t)v(t)dt = \int_a^b v(t)M_t K(z,t)dt = \int_a^b K(z,t)M^\dagger_t v(t)dt = 0.
\]
The general method is given \( L_z \), find \( M_t \), solve for homogeneous solutions of the adjoint equation with boundary conditions that make the boundary term \( Q(K(z,t)v(t))|_{t=a}^{t=b} \) vanish. Several examples will be given in the material that follows.

**Lecture 28 - Topic 2: Kernels of integral representations**

The last topic provides no guidance on how to construct integral representation. There are some general methods that work for certain kinds of equations. The following are general guidelines:

1. Linear differential equations with coefficients that are linear functions of \( z \) can be solved with the Laplace kernel:
   \[
   K(z,t) = e^{zt}
   \]

2. Linear differential equations where the coefficients of \( \frac{d^n u}{dz^n} \) are polynomials of degree \( n \) can be solved with the Euler kernel:
   \[
   K(z,t) = (z - t)^u
   \]
   where \( u \) is a complex number.

3. Linear differential equations of the form
   \[
z^n H_1(z \frac{d}{dz})u(z) + H_2(z \frac{d}{dz})u(z) = 0
   \]
   where \( H_i \) are functions of \( z \frac{d}{dz} \), can be solved using a Mellin kernel of the form,
   \[
   K(z,t) = F(z')
   \]
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4 Bessel functions use kernels of the form

\[ K(z, t) = \left(\frac{z}{2}\right) e^{t - z^2/4t} \]

The form of operator \( M_t \) and the choice of \( v(t) \) depends on the differential equation.

Lecture 28 - Topic 3: Fuchsian equations with three regular points

In this section we show that class of equations with three singularities can all be related to solutions of an equation called the hypergeometric differential equation. It turns out that many equations that appear in physics problems are related to the hypergeometric function, which is an analytic solution of this equation.

Consider differential equations of the form

\[
\frac{d^2 u(z)}{dz^2} + \left( \frac{1 - \alpha - \alpha'}{z - z_1} + \frac{1 - \beta - \beta'}{z - z_2} + \frac{1 - \gamma - \gamma'}{z - z_3} \right) \frac{du(z)}{dz} + \frac{(z_1 - z_2)(z_1 - z_3)\alpha\alpha'}{z - z_1} + \frac{(z_2 - z_1)(z_2 - z_3)\beta\beta'}{z - z_2} + \frac{(z_3 - z_1)(z_3 - z_2)\gamma\gamma'}{z - z_3} \times \frac{u(z)}{(z - z_1)(z - z_2)(z - z_3)}
\]

where \( \alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1 \).

This equation has regular singularities at \( z = z_1, z = z_2 \) and \( z = z_3 \). The form of the equation is chosen so the indicial equation about each singular point has the form

\[
\begin{align*}
\text{at } z = z_1: & \quad r(r - 1) + (1 - \alpha - \alpha')r + \alpha\alpha' = (r - \alpha)(r - \alpha') = 0 \\
\text{at } z = z_2: & \quad r(r - 1) + (1 - \beta - \beta')r + \beta\beta' = (r - \beta)(r - \beta') = 0 \\
\text{at } z = z_3: & \quad r(r - 1) + (1 - \gamma - \gamma')r + \gamma\gamma' = (r - \gamma)(r - \gamma') = 0
\end{align*}
\]

which shows that the roots are \( \alpha, \alpha', \beta, \beta', \gamma, \) and \( \gamma' \). (You should check this).

This equation is called the equation of Riemann. The solutions are represented by

\[
u(z) = P \begin{pmatrix} z_1 & z_2 & z_2 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix}
\]

where \( z_i \) represent the singular points, followed by the roots of the indicial equation for that point in each column. This is called a P-symbol. As it is defined it represents any solution to the differential equation.
This solution has 9 parameters, \( z_1, z_2, z_3, \alpha, \beta, \gamma \) and \( \alpha', \beta', \gamma' \). This can be reduced to a set of three parameters by defining

\[
v(z) = (z - z_1)^r (z - z_2)^s (z - z_3)^t u(z) \quad r + s + t = 0
\]

\[
v(z) = (z - z_1)^r (z - z_2)^s (z - z_3)^t P \begin{pmatrix} z_1 & z_2 & z_3 \\
\alpha & \beta & \gamma & z \\
\alpha' & \beta' & \gamma' & z \\
\end{pmatrix} \tag{22}
\]

After some algebra this becomes the solution of another equation of the same form, where the roots of the indicial equation about each singularity have shifted

\[
v(z) = P \begin{pmatrix} z_1 & z_2 & z_3 \\
\alpha + r & \beta + s & \gamma + t & z \\
\alpha' + r & \beta' + s & \gamma' + t & z \\
\end{pmatrix}.
\]

We can also make a variable change using the homographic transformation discussed last semester:

\[
z' = \frac{Az + B}{Cz + D}, \quad z'_i = \frac{Az_i + B}{Cz_i + D}.
\]

This transformation shift the position of the singular points and changes the variable of the differential equation:

\[
P \begin{pmatrix} z_1 & z_2 & z_3 \\
\alpha & \beta & \gamma & z \\
\alpha' & \beta' & \gamma' & z \\
\end{pmatrix} = P \begin{pmatrix} z'_1 & z'_2 & z'_3 \\
\alpha & \beta & \gamma & z'_i \\
\alpha' & \beta' & \gamma' & z'_i \\
\end{pmatrix}
\]

which gives another equation of the same form with different regular singular points. The verification of these formulas is tedious but uses only algebra and differentiation.

We can use both of these transformations to express the solution for any \( P \) symbol in terms of a corresponding solution of a standard \( P \)-symbol with only three parameters. The first step is to use the homographic transformation to transform the singular points to \( z'_1 = 0, z'_2 = \infty, z'_3 = 1 \). This requires

\[
\begin{align*}
A &= \frac{(z_3 - z_2)}{z_2(z_1 - z_3)} \\
B &= \frac{z_1(z_2 - z_3)}{z_2(z_1 - z_3)} \\
C &= -\frac{1}{z_2}
\end{align*}
\]

which gives the variable transformation:

\[
z' = \frac{(z_3 - z_2)(z - z_1)}{(z_3 - z_1)(z - z_2)}
\]

and the relation

\[
P \begin{pmatrix} z_1 & z_2 & z_3 \\
\alpha & \beta & \gamma & z \\
\alpha' & \beta' & \gamma' & z \\
\end{pmatrix} = P \begin{pmatrix} 0 & \infty & 1 \\
\alpha & \beta & \gamma & z' \\
\alpha' & \beta' & \gamma' & z' \\
\end{pmatrix}.
\]
This replaces the 9 parameter expression by a 6 parameter expression. Next we use the transformation (22) with
\[ r = -\alpha \quad s = \alpha + \gamma \quad t = -\gamma \]
which gives
\[
P \begin{bmatrix} z_1 & z_2 & z_2 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{bmatrix} = \left( \frac{z - z_1}{z - z_2} \right)^\alpha \left( \frac{z - z_3}{z - z_2} \right)^\gamma P \begin{bmatrix} 0 & \infty & 1 \\ 0 & a & 0 \\ 1 - c & b & c - a - b \end{bmatrix} z' \]
where
\[ a = \alpha + \beta + \gamma \quad b = \alpha' + \beta' + \gamma \quad c = 1 + \alpha - \alpha' \]
The key result is that any \( P \) symbols can be expressed in terms of the following 3-parameter \( P \) symbol:
\[
P \begin{bmatrix} 0 & \infty & 1 \\ 0 & a & 0 \\ 1 - c & b & c - a - b \end{bmatrix}. \tag{23} \]
Making these substitutions in the original differential equation, the equation satisfied by (23) becomes
\[
z(z - 1) \frac{d^2 u}{dz^2} + (c - (a + b + 1)z) \frac{du}{dz} - abu(z) = 0 \]
This equation is called the \textit{hypergeometric equation}. It still has 2 independent solutions. The solution that is analytic in a neighborhood of the origin is denoted by \( F(a, b, c; z) \). It is called the \textit{hypergeometric function}.

Putting everything together shows that one solution of a general symbol is related to the hypergeometric function as follows
\[
P \begin{bmatrix} z_1 & z_2 & z_2 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{bmatrix} = \left( \frac{z - z_1}{z - z_2} \right)^\alpha \left( \frac{z - z_3}{z - z_2} \right)^\gamma F \left( \alpha + \beta + \gamma, \alpha + \beta' + \gamma, 1 - \alpha - \alpha'; \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_2)} \right) \]
Even the hypergeometric differential equation will have 2 independent solutions in the neighborhood of each singularity. The relation of these to \( F(a, b, c; z) \) will be discussed in the next topic.

\textbf{Lecture 28 - Topic 4: The Hypergeometric equation}

This topic is concerned with solutions of the hypergeometric equation:
\[
z(z - 1) \frac{d^2 u(z)}{dz^2} + (c - (a + b + 1)z) \frac{du(z)}{dz} - abu(z) = 0 \tag{24} \]
This equation has singularities at 0, 1 and $\infty$. In general the solutions will be multi-valued. We choose a branch cut running from $1 \to \infty$ along the positive real axis. The analytic solution in the neighborhood of $z_0 = 0$ has the form

$$F(a, b, c; z) = \sum_{n=0}^{\infty} f_n z^n$$

where substituting the series in (24) gives the following equations for the expansion coefficients $f_n$

$$f_n = \frac{(a + n - 1)(b + n - 1)}{n(c + n - 2)} f_{n-1} \quad f_0 := 1$$

This gives a convergent series provided $c$ is not a non-negative integer. In that case the solution is represented by the series

$$F(a, b, c; z) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} z^n.$$ 

Since there is pole at $z = 1$ this series converges uniformly for $|z| < 1$.

When $c$ is not a non-negative integer, then the second solution has the form

$$u_2(z) = z^r g(z) = z^{1-c} g(z).$$

Substituting this in (24)

$$z(z-1)\frac{d^2 u(z)}{dz^2} + (c - (a + b + 1)z) \frac{du(z)}{dz} - abu(z) = 0$$

after performing algebra, gives the following differential equation for $g(z)$:

$$z(z-1)\frac{d^2 g(z)}{dz^2} + (c - 2 + (a + b - 2c + 3)z) \frac{dg(z)}{dz} + (a - c + 1)(b - c + 1)g(z) = 0$$

This is the hypergeometric equation for

$$F(b - c + 1, a - c + 1, 2 - c, z).$$

It follows that an independent second solution of the hypergeometric differential equation in a neighborhood of the origin is

$$u_2(z) = z^{1-c}F(b - c + 1, a - c + 1, 2 - c, z).$$

This one generally has a branch cut due to the factor $z^{1-c}$.

In general there are two solutions associated with each of the three regular singular points. For the next topic we show how they are also related to $F(a, b, a; z)$

Lecture 29 - Topic 1: The Hypergeometric equation
In the previous lecture we constructed two independent solutions of the hypergeometric differential equation. One was analytic in a neighborhood of the origin and the other had an isolated singularity at $z = 0$, but both could be expressed in terms of a solution analytic in a neighborhood of the origin (the hypergeometric function).

There are also solutions of the equation in the neighborhood of the other two singular points. The differential equation associated with a given $P$ symbol remains unchanged under exchanging the columns of the $P$ symbol and invariant under the interchange of the roots of the indicial equation $\alpha \leftrightarrow \alpha'$, $\beta \leftrightarrow \beta'$, $\gamma \leftrightarrow \gamma'$. There are $48 = 3 \times 2^3$ such permutations that leave the equation unchanged.

While the $P$-symbol

$$P \left\{ \begin{array}{ccc} z_1 & z_2 & z_2 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{array} \right\}$$

has these symmetries, the solution

$$\left( \frac{z - z_1}{z - z_2} \right)^\alpha \left( \frac{z - z_1}{z - z_2} \right)^\gamma F \left( \alpha + \beta + \gamma, \alpha + \beta' + \gamma, 1 - \alpha - \alpha'; \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_2)} \right)$$

does not have these symmetries. For the hypergeometric differential equation $z_i$ are $\{0, 1, \infty\}$ and the roots of the indicial equations are $\alpha = 0$, $\alpha' = 1 - c$, $\beta = a$, $\beta' = b$, $\gamma = 0$ and $\gamma' = c - a - b$, but the solution above not have these symmetries, although it is invariant under $a \leftrightarrow b$. These transformations generate $48/2 = 24$ solutions of the hypergeometric differential equation that can all be expressed directly in terms of the hypergeometric function. The $48 \to 24$ is due to the symmetry $a \leftrightarrow b$ of this solution.

Without going into the details, these relations can be used to express the analytic and singular solutions of the hypergeometric equation about the other singular points, $z_0 = 1$ and $z_0 = \infty$ in terms of the hypergeometric function. The expressions are

$$G_{11}(z) = F(a, b, a + b + 1 - c; 1 - z)$$
$$G_{12}(z) = (1 - z)^{c-a-b} F(c - b, c - a, 1 + c - a - b; 1 - z)$$
$$G_{\infty 1}(z) = z^{-a} F(a, a - c + 1, a - b + 1; \frac{1}{z})$$
$$G_{\infty 2}(z) = z^{-b} F(b, b - c + 1, b - a + 1; \frac{1}{z})$$

The important observation is that all of the solutions of interest can be expressed directly in terms of the hypergeometric function.

**Lecture 29 - Topic 2: Integral representations of the hypergeometric function**
The radius of convergence of the series expansion of the hypergeometric function about $z = 0$ is 1 due to the singular point at 1. This can be extended using integral representations.

The hypergeometric equation has the form of equations where an integral representation can be constructed using a kernel of the Euler form:

$$u(z) = \int_C (z - t)^\lambda v(t)dt$$

where we use the hypergeometric equation to determine $\lambda$, $v(t)$ and the contour $C$. The hypergeometric equation is

$$z(1 - z)u''(z) + [c - z(a + b + 1)]u'(z) - abu(z) = 0.$$ 

The first step in the construction is to insert the integral representation into the differential equation. The derivatives are

$$u'(z) = \int_C \lambda(z - t)^{\lambda-1}v(t)dt$$

$$u''(z) = \int_C \lambda(\lambda - 1)(z - t)^{\lambda-2}v(t)dt.$$ 

Using these expressions for the derivatives in the differential equation gives

$$\int_C (z(1 - z)\lambda(\lambda - 1)(z - t)^{\lambda-2} + [c - z(a + b + 1)]\lambda(z - t)^{\lambda-1} - ab(z - t)^\lambda) v(t)dt = 0.$$ 

Next we factor out $(z - t)^{\lambda-2}$ to get

$$\int_C (z(1 - z)\lambda(\lambda - 1) + [c - z(a + b + 1)]\lambda(z - t) - ab(z - t)^2) (z - t)^{\lambda-2}v(t)dt = 0.$$ 

Next we express the coefficient as a polynomial in powers of $z$

$$0 = \int_C (z^2(-\lambda^2 + \lambda - \lambda(a + b + 1) - ab) + z(\lambda^2 - \lambda + c\lambda + t\lambda(a + b + 1) + 2abt) + (-c\lambda t - abt^2)) (z - t)^{\lambda-2}v(t)dt$$

We are now in a position to determine $\lambda$. We choose it so the coefficient of $z^2$ is zero. This gives a quadratic polynomial for $\lambda$

$$-\lambda^2 + \lambda - \lambda(a + b + 1) - ab = 0$$

or

$$\lambda^2 + \lambda(a + b) + ab = 0$$
which has roots
\[ \lambda = -a; \quad \lambda = -b. \]
We can choose either root - each choice is associated with a different integral representation. In what follows we make the choice \( \lambda = -a \) which leads to the equations
\[
0 = \int_C \left( z(a^2 + a - ca - ta(a + b + 1) + 2abt) + (ca - abt^2) \right) (z - t)^{-a-2} v(t) dt
\]
\[
\int_C \left( z(a^2 + a - ca - ta(a - b + 1)) + (ca - abt^2) \right) (z - t)^{-a-2} v(t) dt
\]
Next we replace \( z = z - t + t \) to get
\[
\int_C \left( (z - t)(a^2 + a - ca - ta(a - b + 1)) + t(a^2 + a - ta(a + 1)) (z - t)^{-a-2} v(t) dt =
\]
\[
\int_C \left( (z - t)a(a + 1 - c - t(a - b + 1)) + t(1 - t)a(a + 1) (z - t)^{-a-2} v(t) dt =
\]
\[
\int_C \left( -(z - t)a(-a - 1 + c + t(a - b + 1)) - t(1 - t)a(-a - 1) (z - t)^{-a-2} v(t) dt.
\]
This can be expressed as a differential operator in \( T \)
\[
0 = \int_C \left( (a + 1 - c + t(b - a - 1)) \frac{d}{dt} + t(1 - t) \frac{d^2}{dt^2} \right) (z - t)^{-a} v(t) dt.
\]
We define the differential operator
\[
M_t = t(1 - t) \frac{d^2}{dt^2} + (a + 1 - c + t(b - a - 1)) \frac{d}{dt}
\]
Following what was discussed in topic 2 of lecture 28 we want \( v(t) \) to be a solution of the adjoint equation which has the form:
\[
0 = M^+ v(t) = \frac{d^2}{dt^2} t(1 - t) v(t) - \frac{d}{dt} (a + 1 - c + t(b - a - 1)) v(t) = 0.
\]
Integrating once gives
\[
\frac{d}{dt} t(1 - t) v(t) - ((a + 1 - c + t(b - a - 1)) v(t) = 0.
\]
To integrate this let
\[
w(t) := t(t - 1) v(t)
\]
which gives
\[
\frac{w'(t)}{w(t)} = -\frac{(a + 1 - c + t(b - a - 1))}{t(t - 1)}.
\]
Integrating once again gives
\[
\ln w(t) = -\int (a + 1 - c)(-\frac{1}{t} + \frac{1}{t - 1}) - \int \frac{b - a - 1}{t - 1}
\]
which gives
\[
\ln w(t) = (a + 1 - c) \ln(t) + (-1 - a + c - b + a + 1) \ln(t - 1)
\]
\[
w(t) = Ct^{a+1-c}(t-1)^{c-b}
\]
\[
v(t) = Ct^{a-c}(t-1)^{c-b-1}
\]
where \(C\) is a constant.

Now that we have and expression for \(v(t)\) we use the Lagrange identity to determine the boundary conditions that make the surface term vanish. To do this consider the difference
\[
v(t)M_t(z - t)^{-a} - (z - t)^{-a}M_t^tv(t)
\]
using the expressions for \(v(t)\) after some algebra gives the following expression for the boundary term:
\[
v(t)M_t(z - t)^{-a} - (z - t)^{-a}M_t^tv(t) = \frac{d}{dt}aCt^{a-c+1}(t-1)^{c-b}(z - t)^{-a-1}.
\]
This vanishes for \(t = 1\) and \(t = \infty\) provided
\[
Re(c) > Re(b) > 0
\]
The next step is to choose a contour between the two points. One choice is to directly integrate along the real axis from 1 to \(\infty\). Putting everything together we get an integral representation of the hypergeometric function
\[
F(a, b; c; z) = C \int_1^\infty dt(t - z)^{-a}t^{a-c}(t - 1)^{c-b-1}
\]
It remains to find the multiplicative constant \(C\). Here we changed the constant to express this in terms of \(t - z\) rather than \(z - t\). This is because \(t > 1\) and we want to compare to the power series expansion of \(F(a, b; c; z)\) for \(|z| < 1\)

Expanding \((t - z)^{-a}\) in powers of \(z\) using
\[
(t - z)^{-a} = t^{-a} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)}{\Gamma(a)\Gamma(n + 1)}(\frac{z}{t})^n
\]
and using this expansion in the integral representation gives
\[
F(a, b; c; z) = C \int_1^\infty dt(z - t)^{-a}t^{a-c}(t - 1)^{c-b-1} =
\]
\[ \int_1^\infty dt \frac{1}{t^a - t^c} = -c - n \left( \frac{t}{c - b} - 1 \right) \]

where the integral is the beta function (see page 96 equation 32.8 of the text)

\[ = C \sum_{n=0}^\infty z^n \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)\Gamma(n+1)} \int_1^\infty dt \frac{t^{-c-n}(t-1)^{c-b-1}}{\Gamma(c)} \]

where this has the form of a constant times the series expansion of \( F(a, b, x; z) \):

\[ = C \frac{\Gamma(c-b)\Gamma(b)}{\Gamma(c)} F(a, b, c; z) \]

This allows us to calculate the coefficient \( C \):

\[ C = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \]

\[ F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_1^\infty dt (z-t)^{-a}t^{a-b-1}(t-1)^{c-b-1} \]

For homework I will ask you to make the variable \( t \to 1/t \) in the integral representation to show \( F(a, b, c; z) \) can also be represented by

\[ F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt (1-zt)^{-a}t^{a-b-1}(1-t)^{c-b-1} \]

which converges for \( \text{Re}(c) > \text{Re}(b) > 0 \). This is called the Euler representation.

**Lecture 29 - Topic 3: Relations**

There are a number of identities associated with hypergeometric functions. If we start with the series representation of the hypergeometric function

\[ F(a, b, c; z) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^\infty \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} z^n. \]

(25)

and differentiate \( n \) times we get

\[ \frac{d^n F(a, b, c; z)}{dx^n} := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^\infty \frac{\Gamma(a+m+n-m)\Gamma(b+m+n-m)}{\Gamma(c+m+n-m)(n-m)!} z^{n-m} = \]

\[ \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(c+m)} F(a+m, b+m, c+m; z) \]
The functions $F(a \pm 1, b, c; z)$, $F(a, b \pm 1, c; z)$ and $F(a, b, c \pm 1; z)$ are called hypergeometric functions \textit{contiguous} to $F(a, b, c; z)$. There are a number of relations between these functions that can be derived by comparing coefficients of the power series. Two examples are (25):

\[(c - 2a - (b - a)z)F(a, b, c; z) + a(1 - z)F(a + 1, b, c; z) - (c - a)F(a - 1, b, c; z) = 0\]

\[(c - a - 1)F(a, b, c; z) + aF(a + 1, b, c; z) - (c - 1)F(a, b, c - 1; z) = 0.\]

We illustrate the last one. Expressed in term of the series it has the form

\[
(c - a - 1) \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)n!} z^n + a \frac{\Gamma(c)}{\Gamma(a + 1)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a + 1 + n)\Gamma(b + n)}{\Gamma(c + n)n!} z^n - (c - 1) \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n - 1)n!} z^n.
\]

The coefficient of $z^n$ is

\[
\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \left( (c - a - 1) \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)n!} + a \frac{\Gamma(a + 1 + n)\Gamma(b + n)}{\Gamma(c + n)n!} - (c - 1) \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n - 1)n!} \right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)n!} \left( (c - a - 1) + \frac{a}{a} (a + n) - \frac{c - 1}{c - 1} (c - n - 1) \right) = 0.
\]

There are a total of 15 such relations involving contiguous functions.

Another identity uses the Euler integral representation mentioned at the end of the last topic.

\[
F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 dt (1 - zt)^{-a} t^{b-1} (1 - t)^{c-b-1}.
\]

Change variables so $t' = 1 - t$ gives

\[
= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 dt (1 - z + zt)^{-a} (1 - t)^{b-1} t^{c-b-1} = \frac{\Gamma(c)(1 - z)^{-a}}{\Gamma(b)\Gamma(c - b)} \int_0^1 dt (1 + \frac{zt}{1 - z})^{-a} (1 - t)^{b-1} t^{c-b-1} = (1 - z)^{-a} F(a, c - b, c, \frac{z}{z - 1}).
\]

\[
F(a, b, c; z) = (1 - z)^{-a} F(a, c - b, c, \frac{z}{z - 1}).
\]

These analytic functions agree when $|z| < |z - 1| < 1$ and $c$ is not 0 or a negative integer. This integral representation extend the domain of analyticity to a much larger region.
Using the representation for the beta function again we get the following normalization:

\[
F(a, b, c; 1) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c - b)} \int_0^1 dt (1 - t)^{-a} t^{b-1} (1 - t)^{c-b-1} =
\]

\[
\frac{\Gamma(c)}{\Gamma(b) \Gamma(c - b)} \int_0^1 dt t^{b-1} (1 - t)^{c-b-1-a} =
\]

\[
\frac{\Gamma(c) \Gamma(b) \Gamma(c - b - a)}{\Gamma(c - b) \Gamma(c - a)} = \frac{\Gamma(c) \Gamma(c - b - a)}{\Gamma(c - b) \Gamma(c - a)}
\]

We have determined how to express the independent solutions in the neighborhood of \(z = 1\)

\[
G_{11}(z) = F(a, b, a + b + 1 - c; 1 - z)
\]

\[
G_{12}(z) = (1 - z)^{c-a-b} F(c - b, c - a, 1 + c - a - b; 1 - z)
\]

In the region where the domains overlap, \(F(a, b, c; z)\) should be a linear combination of these two functions

\[
F(a, b, c; z) = \alpha F(a, b, a + b + 1 - c; 1 - z) + \beta (1 - z)^{c-a-b} F(c - b, c - a, 1 + c - a - b; 1 - z)
\]

The coefficients can be found by setting \(z = 0\) and \(z = 1\). This gives the relations:

\[
F(a, b, c; 1) = \alpha F(a, b, a + b + 1 - c; 0) + \beta (1 - z)^{c-a-b} F(c - b, c - a, 1 + c - a - b; 1)
\]

Using

\[
F(a, b, c; 0) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)}
\]

and

\[
F(a, b, c; 1) = \frac{\Gamma(c) \Gamma(c - b - a)}{\Gamma(c - b) \Gamma(c - a)}
\]

in the above equation and solving for \(\alpha\) and \(\beta\) gives

\[
\alpha = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} \quad \beta = \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)}
\]

which gives

\[
F(a, b, c; z) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} F(a, b, a + b + 1 - c; 1 - z) +
\]

\[
(1 - z)^{c-a-b} \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} F(c - b, c - a, 1 + c - a - b; 1 - z)
\]

This provides an analytic continuation of \(F(a, b, c; z)\) in a neighborhood of \(z = 1\). There are many more relations that can be derived among the solutions of the hypergeometric differential equation.
Lecture 29 - Topic 4: Relation to special functions

Some of the equations satisfied by the classical orthogonal polynomials are the Jacobi differential equation

\[(1 - z)^2 \frac{d^2 u(z)}{dz^2} + (\beta - \alpha(z + \beta + 2)z) \frac{du(z)}{dz} + \lambda(\lambda + \alpha + \beta + 1)u(z) = 0\]

and the Gegenbauer equation

\[(1 - z)^2 \frac{d^2 u(z)}{dz^2} + (2\mu + 1)z \frac{du(z)}{dz} + \lambda(\lambda + 2\mu)u(z) = 0.\]

For the classical polynomials, \(\lambda = n\) is an integer. Making the substitution \(z = 1 - 2x \rightarrow 1 - 2z\) transforms these to equations of the hypergeometric type

\[z(1 - z)u''(z) + [c - z(a + b + 1)]u'(z) - abu(z) = 0.\]

The first one is satisfied by

\[P_{\lambda}^{(\alpha,\beta)}(z) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + \alpha + \beta + 1)} F(-\lambda, \lambda + \alpha + \beta + 1, \alpha + 1, 1 - \frac{z}{2})\]

This is called the Jacobi function of the first kind. The second one is satisfied by

\[G_{\lambda}^{(\mu)}(z) = \frac{\Gamma(2\mu + \lambda)}{\Gamma(\lambda + (2\mu))} F(-\lambda, \lambda + 2\mu, \mu + 1, 1 - \frac{z}{2}).\]

Both of these solutions become polynomials when \(\lambda = n\) is a non-negative integer. To show this we take the limit \(\lambda \rightarrow n\) in the series solution. For example consider

\[\lim_{\lambda \rightarrow m} G_{\lambda}^{(\mu)}(z) = \lim_{\lambda \rightarrow m} \frac{\Gamma(2\mu + \lambda)}{\Gamma(\lambda + (2\mu))} F(-\lambda, \lambda + 2\mu, \mu + 1, 1 - \frac{z}{2})\]

\[\lim_{\lambda \rightarrow m} \frac{\Gamma(2\mu + \lambda)}{\Gamma(\lambda + (2\mu))} \frac{\Gamma(\mu + \frac{1}{2})}{\Gamma(-\lambda)\Gamma(\lambda + 2\mu)} \sum_{n=0}^{\infty} \frac{\Gamma(-\lambda + n)\Gamma(\lambda + 2\mu + n)}{\Gamma(\mu + \frac{1}{2} + n)n!} z^n\]

Properties of the Gamma function give the factor

\[\lim_{\lambda \rightarrow m} \frac{\Gamma(-\lambda + n)}{\Gamma(-\lambda)} = \begin{cases} (-)^n n! & n \leq m \\ 0 & n > m \end{cases}\]

eliminates the terms for \(n > m\). The same mechanism makes the Jacobi function of the first kind become a polynomial when \(\lambda = n \geq 0\)

These are the analytic solutions of these equations - the other independent solutions of these equations are not analytic. They are also related to the hypergeometric functions by

\[Q_{\lambda}^{(\alpha,\beta)}(z) = \frac{2^{\lambda+\alpha+\beta}\Gamma(\lambda + \alpha + 1)\Gamma(\lambda + \beta + 1)}{\Gamma(2\lambda + \alpha + \beta + 2)(\alpha + 1)(z + 1)^{\lambda+\alpha+1}} F(\lambda + 1, \lambda + \alpha + 1, 2\lambda + \alpha + \beta + 2; \frac{2}{1 - z}).\]
This solution is called the Jacobi function of the second kind. There is also a
not analytic solution independent solution of the Gegenbauer equations. It is
proportional to a particular Jacobi function of the second kind:

\[ Q_{\lambda}^{\mu-\frac{1}{2},\mu-\frac{1}{2}}(z) \]

The most important special case of the Jacobi functions are the case \( \alpha = \beta = 0 \)
which give the Legendre polynomials. The Legendre function of the first
kind is

\[ P_{\lambda}(z) = P_{\lambda}^{(0,0)}(z) \]

which becomes a polynomial when \( \lambda = n \) is a non-negative integer.

The second independent non-analytic solution is called the Legendre function
of the second kind. It is related to the Jacobi function of the second kind by

\[ Q_{\lambda}(z) = Q_{\lambda}^{(0,0)}(z) \]

The important observation is how all these solutions re related to the hyper-
geometric function,

\[ F(a,b,c;z) \]

Lecture 30 - Topic 1: The confluent hypergeometric function

We start by considering the special case of the Riemann equation:

\[
\frac{d^2 u(z)}{dz^2} + \left( \frac{c}{z} + \frac{1-a-b}{z-z_2} + \frac{1-c-a+b}{z-z_3} \right) \frac{du(z)}{dz} + \frac{abz_2(z_2-z_3)}{z(z-z_2)^2(z-z_3)} u(z) = 0
\]

This has regular singular points at \( z = 0, z_2 \) and \( z_3 \). Let \( z_2 = 2z_3 = b \) and take
the limit as \( b \to \infty \), keeping \( a \) and \( c \) fixed.

In this limit the Riemann equation reduces to

\[
\frac{d^2 u(z)}{dz^2} + \left( \frac{c}{z} + 1-2 \right) \frac{du(z)}{dz} - \frac{au(z)}{z} = 0
\]

Multiplying by \( z \) gives

\[
z \frac{d^2 u(z)}{dz^2} + (c-z) \frac{du(z)}{dz} - au(z) = 0.
\]

This is called the confluent Riemann equation. This is a limiting form of the
hypergeometric equation where the singularity at 1 is moved out to infinity. In
this limit \( z = \infty \) is no longer a regular singular point. To see this let \( w = 1/z \),
then

\[
\frac{d}{dz} = \frac{dw}{dz} \frac{d}{dw} = -\frac{1}{z^2} \frac{d}{dw} = -w^2 \frac{d}{dw}
\]

and

\[
\frac{d^2}{dz^2} = (w^2 \frac{d}{dw})(w^2 \frac{d}{dw}) = w^4 \frac{d^2}{dw^2} + 2w^3 \frac{d}{dw}.
\]
Using these in the differential equation gives
\[ \frac{1}{w}(w^4 \frac{d^2 u}{dw^2} + 2w^3 \frac{du}{dw} - w^2(c - \frac{1}{w} \frac{du}{dw}) - au = 0 \]
Dividing by \( w^3 \) gives
\[ \frac{d^2 u}{dw^2} + \left( \frac{2 - c}{w} + \frac{1}{w^2} \right) \frac{du}{dw} - \frac{a}{w^3} u = 0 \]
which shows that the coefficients of both \( u \) and \( \frac{du}{dw} \) are too singular at the origin, \((w = 0 \rightarrow z = \infty)\), to be considered regular.

The indicial equation for the confluent Riemann equation equation is
\[ 0 = r(r - 1) + cr = r(r + c - 1) \]
which has roots \( r = 0 \) and \( r = 1 - c \). The solution that is analytic in the neighborhood of the origin is denoted by
\[ \Phi(a, c; z) \]
with normalization
\[ \Phi(a, c; 0) = 1. \]
Since the only other singularities in the equation are at \( z = \infty \) \( \Phi(a, c; z) \) is an entire function called the **confluent hypergeometric function**.

We will construct this solution using the method of integral representations again. In this case we use the Laplace kernel and write
\[ u(z) = \int_C e^{zt}v(t)dt. \]
The first step is to determine the differential operator \( M_t \).
\[ L_zu(z) = \int_C (zt^2 + (c - z)t - a)e^{zt}v(t)dt. \]
Next we replace the \( z \) dependence by a differential operator in \( t \):
\[ L_zu(z) = \int_C v(t)(t^2 \frac{d}{dt} + t(c - \frac{d}{dt}) - a)e^{zt}dt = \int_C v(t) \left( (t^2 - t) \frac{d}{dt} + (tc - a) \right) e^{zt}dt. \]
Integrating by parts gives the adjoint operator
\[ \int_C e^{zt} \left( -\frac{d}{dt}(t^2 - t) + (tc - a) \right) v(t)dt \]
where we have to choose \( C \) so there are no boundary terms. To do this we first have to find solutions \( v(t) \) of the adjoint equation. To find homogeneous solutions to the adjoint equation we integrate

\[- (t^2 - t)v' - (2t - 1 + a - ct)v = 0\]

which can be expressed as

\[\frac{v'}{v} = -(\frac{2 - c}{t - 1} + (1 - a)(-\frac{1}{t} + \frac{1}{t - 1})].\]

Integrating

\[\ln(v) = \ln(t - 1)^{c-1-a} + \ln(t)^a - 1\]

\[v = C(t - 1)^{c-a-1}t^a - 1\]

where \( C \) is a constant. The boundary terms that must vanish when integrating by parts are

\[e^{zt}(t^2 - t)v(t) = e^{zt}(t^2 - t)C(t - 1)^{c-a-1}t^a - 1 = C e^{zt}(t - 1)^{c-a}t^a.\]

The path of integration must be chosen so that either vanish at the endpoints or have identical values at the endpoints. If \( \text{Re}(c) > \text{Re}(a) > 0 \) then this vanishes at \( t = 0 \) and \( t = 1 \). This results in the integral representation

\[u(z) = C \int_0^1 e^{zt}t^{a-1}(1 - t)^{c-a-1}dt\]

where we have changed \( C \) to change \((t - 1)\) to \((1 - t)\). To find the normalization condition we expand the exponential term in powers of \( z \):

\[u(z) = C \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_0^1 t^{a-1+n}(1 - t)^{c-a-1}dt\]

where the integral is \( B(a + n, c - a) \) which gives

\[u(z) = C \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma(a + n)\Gamma(c - a)}{\Gamma(n + 1)\Gamma(c + n)}\]

\[1 = u(0) = C \frac{\Gamma(a)\Gamma(c - a)}{\Gamma(c)}\]

or

\[C = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)}\]

This gives the integral representation and the series representation of

\[\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 e^{zt}t^{a-1}(1 - t)^{c-a-1}dt = \]

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\[
\frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} z^n \frac{\Gamma(a+n)}{\Gamma(n+1)\Gamma(c+n)}.
\]

Comparing the expansion with the corresponding expansion for the hypergeometric functions

\[
\lim_{b \rightarrow \infty} F(a, b, c, \frac{z}{b}) = \lim_{b \rightarrow \infty} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} z^n \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(n+1)b^n\Gamma(c+n)} =
\]

\[
\frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} z^n \frac{\Gamma(a+n)}{\Gamma(n+1)\Gamma(c+n)} = \Phi(a, c; z)
\]

where we have used

\[
1 = \lim_{b \rightarrow \infty} \frac{\Gamma(b+n)}{b^n\Gamma(b)} = \lim_{b \rightarrow \infty} (1 + \frac{n-1}{b})(1 + \frac{n-2}{b}) \cdots (1 + \frac{1}{b})1.
\]

This gives the relation of the confluent hypergeometric function to the hypergeometric function:

\[
\Phi(a, c; z) = \lim_{b \rightarrow \infty} F(a, b, c, \frac{z}{b}).
\]

We get another integral representation of \(\Phi(a, c; z)\) using the above limit in the Euler integral representation of the hypergeometric equation

\[
\Phi(a, c; z) = \lim_{b \rightarrow \infty} F(a, b, c, \frac{z}{b}) = \lim_{b \rightarrow \infty} F(b, a, c, \frac{z}{b}) =
\]

\[
\lim_{b \rightarrow \infty} \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_{0}^{1} dt \left(1 - \frac{tz}{b}\right)^{-b} t^{a-1} (1 - t)^{c-a-1}
\]

\[
\lim_{b \rightarrow \infty} \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_{0}^{1} dt e^{zt} t^{a-1} (1 - t)^{c-a-1}
\]

where we have used the symmetry of hypergeometric function on interchanging \(a\) and \(b\) and

\[
\lim_{b \rightarrow \infty} (1 - \frac{tz}{b})^{-b} = e^{zt}
\]

which can be understood by writing

\[
\lim_{b \rightarrow \infty} (1 - \frac{tz}{b})^{-b} = (1 + \frac{tz}{b})^{b} = e^{zt}.
\]

This integral representation is valid for \(\text{Re}(c) > \text{Re}(a) > 0\).

There is also a second non-analytic solution to the hypergeometric differential equation.

Recall when \(c\) is not an integer the non-analytic solution of the hypergeometric equation in a neighborhood of 0 is

\[
z^{1-c} F(b-c+1, a-c+1, 2-c; z)
\]
The second independent solution of the hypergeometric equation is obtained by taking the same limit of this equation
\[
\lim_{b \to \infty} z^{1-c} F\left(b-c+1, a-c+1, 2-c; \frac{z}{b}\right) = \lim_{b \to \infty} z^{1-c} F\left(a-c+1, b-c+1, 2-c; \frac{z}{b}\right) = z^{1-c} \Phi(a-c+1, 2-c, z).
\]

There is another independent solution. Previously we derived the integral representation
\[
u(z) = C \int e^{zt} t^{a-1} (1-t)^{c-a-1} dt
\]
with boundary term
\[
Ce^{zt} (t-1)^{c-a-1}.
\]
This vanishes at 0 and \(-\infty\) is \(Re(a) > 0\) and \(Re(z) > 0\). In this case we get the integral representation
\[
u(z) = C \int_{-\infty}^{0} e^{zt} t^{a-1} (1-t)^{c-a-1} dt.
\]
Changing \(t \to -t\) gives
\[
\Phi(a, c; z) := \frac{1}{\Gamma(a)} \int_{0}^{\infty} e^{-zt} t^{a-1} (1+t)^{c-a-1} dt.
\]
This solution can be expressed as a linear combination of
\[
\Phi(a, c; z) = \alpha \Phi(a, c; z) + \beta z^{1-c} \Phi(a-c+1, 2-c, z).
\]
The coefficients can be computed by comparing the expressions at 2 values. The result is
\[
\Phi(a, c; z) = \frac{(c-1)}{\Gamma(a-c+1)} \Phi(a, c; z) + \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c} \Phi(a-c+1, 2-c, z)
\]

Lecture 30 - Topic 2: Functions related to the confluent hypergeometric function

Parabolic cylinder functions, Hermite functions, Laguerre functions

One of the reasons for studying the confluent hypergeometric functions is because it is related to so many of the standard functions that appear in theoretical physics.

The first class are solutions of the following differential equation
\[
\frac{d^2 u(z)}{dz^2} + \left(\nu + \frac{1}{2} - \frac{z^2}{4}\right) u(z) = 0.
\]
This is called the Weber-Hermite differential equation. We make the substitution
\[
u(z) = e^{-\frac{1}{2}z^2} \tilde{u}(z).
\]
This gives
\[ u'(z) = e^{-\frac{1}{4}z^2} \left( \tilde{u}'(z) - \frac{z}{2} \tilde{u}(z) \right) \]
\[ u''(z) = e^{-\frac{1}{4}z^2} \left( \tilde{u}''(z) - \frac{1}{2} \tilde{u}(z) - \frac{z}{2} \tilde{u}'(z) - \frac{z^2}{2} \tilde{u}(z) + \frac{z^2}{4} \tilde{u}'(z) \right) = \]
\[ u''(z) = e^{-\frac{1}{4}z^2} \left( \tilde{u}''(z) - \frac{1}{2} \tilde{u}(z) - z \tilde{u}'(z) + \frac{z^2}{4} \tilde{u}(z) \right). \]

Using these expressions in the differential equation leads to the following differential equation for \( \tilde{u}(z) \):
\[ 0 = \tilde{u}''(z) - \frac{1}{2} \tilde{u}(z) - z \tilde{u}'(z) + \frac{z^2}{4} \tilde{u}(z) + (\nu + 1/2) \tilde{u}(z) = \]
\[ \tilde{u}''(z) - z \tilde{u}'(z) + \nu \tilde{u}(z) = 0 \]

The next step is to relate this to the solution of the confluent hypergeometric equation. It turns out that \( \tilde{u}(z) \) is related to \( u(z^2/2) \) where \( u(z^2/2) \) is a confluent hypergeometric function.

Note that
\[ \frac{du(z^2/2)}{dz} = z u'(z^2/2) \]
\[ \frac{d^2 u(z^2/2)}{dz^2} = u'(z^2/2) + z^2 u''(z^2/2) \]

using these expression in the equation
\[ \tilde{u}''(z) - z \tilde{u}'(z) + \nu \tilde{u}(z) = 0 \]

give
\[ z^2/2u''(z^2/2) + (1/2)u'(z^2/2) - z^2/2u'(z^2/2) + \nu/2u(z^2/2) \]

Comparing this to the confluent hypergeometric differential equation
\[ zu''(z) + (c - z) u'(z) - au(z) = 0 \]

we find that \( \tilde{u}(z) \) is a solution to the hypergeometric equation with \( c = 1/2 \) and \( a = -\nu/2 \).

Combining everything together gives the following solution to the Weber-Hermite differential equation, which is called the Parabolic Cylinder function:
\[ D_{\nu u}(z) = 2^{\frac{3}{2}} e^{-z^2/4} \Psi \left( -\frac{\nu}{2}, \frac{1}{2}, \frac{z^2}{2} \right) \]

where the \( 2^{\frac{3}{2}} \) is a standard normalization. It can be expressed in terms of the confluent hypergeometric function using the identity from the last topic:
\[ \Phi(a, c; z) = \frac{\Gamma(c - 1)}{\Gamma(a - c + 1)} \Phi(a, c; z) + \frac{\Gamma(c - 1)}{\Gamma(a)} z^{1-c} \Phi(a - c + 1, 2 - c, z) \]
which gives

$$\Psi\left(-\frac{\nu}{2}, \frac{1}{2}; \frac{z^2}{2}\right) =$$

$$\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1+\nu}{2}\right)} \Phi\left(-\frac{\nu}{2}, \frac{1}{2}; \frac{z^2}{2}\right) + \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{\nu}{2}\right)} \Phi\left(1-\nu, \frac{3}{2}; \frac{z^2}{2}\right)$$

(26)

Note that this function is analytic at 0.

Note that when $a = -m$ the confluent hypergeometric function is a polynomial. To see this consider the series expansion of

$$\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{z^n \Gamma(a+n)}{\Gamma(n+1)\Gamma(c+n)}$$

when $a = -m$ is a negative integer

$$\Phi(-m, c; z) = \frac{\Gamma(c)}{\Gamma(-m)} \sum_{n=0}^{\infty} \frac{z^n \Gamma(n-m)}{\Gamma(n+1)\Gamma(c+n)} =$$

$$\sum_{n=0}^{m} (n-m-1)(n-m-2) \cdots (-m)z^n \frac{\Gamma(c)}{\Gamma(n+1)\Gamma(c+n)}$$

which means that $\Phi(-m, c; z)$ is a degree $m$ polynomial.

If we apply this to

$$\Psi\left(-\frac{\nu}{2}, \frac{1}{2}; \frac{z^2}{2}\right)$$

when $n$ is a non-negative integer then using (26) for integer $\nu = m, \nu = 2m$ is even

$$\frac{1}{\Gamma\left(-\frac{2m}{2}\right)} = \frac{1}{\Gamma(-m)} = 0$$

and

$$\Phi\left(-\frac{2m}{2}, \frac{1}{2}; \frac{z^2}{2}\right)$$

$$\Phi(-m, \frac{1}{2}; \frac{z^2}{2})$$

is a polynomial, while if $\nu = 2m + 1$ is odd

$$\frac{1}{\Gamma\left(\frac{1-2m-1}{2}\right)} = \frac{1}{\Gamma(-m)} = 0$$

and

$$\Phi\left(1-\frac{2m+1}{2}, \frac{3}{2}; \frac{z^2}{2}\right)\Phi\left(-m, \frac{3}{2}; \frac{z^2}{2}\right)$$

is a polynomial. In both cases

$$\Psi\left(-\frac{m}{2}, \frac{1}{2}; \frac{z^2}{2}\right)$$
is a polynomial of degree $m$.

The polynomials

$$H_n(z) = 2^n \Psi(-\frac{n}{2}, \frac{1}{2}, \frac{z^2}{2}) = 2^n \frac{z^2}{2} e^{\frac{z^2}{4}} D_n(z)$$

are the Hermite polynomials that we encountered in the context of studying classical orthogonal polynomials.

We showed above that the confluent hypergeometric function for negative integer $a$ are also polynomials. They are up to a multiplicative constant the associated Laguerre polynomials

$$L_n^\mu(z) = \frac{\Gamma(n + \mu + 1)}{\Gamma(n + 1)\Gamma(\mu + 1)} \Phi(-n, \mu + 1, z)$$

Lecture 30 - Topic 3: Functions related to the confluent hypergeometric function

the error function

We use the integral representation

$$\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(c - a)\Gamma(a)} \int_0^1 dt e^{zt} t^{a-1} (1 - t)^{c-a-1}$$

and evaluate

$$z \Phi\left(\frac{1}{2}, \frac{3}{2}; -z^2\right) = \frac{(1/2)\Gamma(1/2)}{\Gamma(1)\Gamma(1/2)} \int_0^1 dt z e^{-z^2 t} t^{1/2} (1 - t)^0$$

Let $x^2 = z^2 t$, $2x dx = z^2 dt$, $t = (x/z)^2$, then this becomes

$$\frac{z}{2} \int_0^z \frac{2x}{z^2} e^{-x^2 z/x} = \int_0^z e^{-x^2} dx = Erf(z)$$

or

$$\boxed{Erf(z) = z \Phi\left(\frac{1}{2}, \frac{3}{2}; -z^2\right)}$$

which is called the error function. It is important in probability theory.

Lecture 30 - Topic 4: Functions related to the confluent hypergeometric function

Bessel functions

Bessel’s equation is

$$b''(z) + \frac{1}{z} b'(z) + \left(1 - \frac{\nu^2}{z^2}\right) b(z) = 0$$
To show how this equation is also related to the confluent hypergeometric equation let
\[ b(z) = ze^{-iz}u(z) \]
and compute
\[ b'(z) = ze^{-iz}(u'(z) + \frac{\nu}{z}u(z) - iu(z)) \]
\[ b''(z) = ze^{-iz} \left( u''(z) + \frac{\nu}{z}u'(z) - \frac{\nu}{z^2}u(z) + \left( \frac{u'}{z}u(z) + \frac{\nu z}{z}u(z) - iu(z) \right) \left( \frac{\nu}{z} - i \right) \right) = \]
\[ ze^{-iz} \left( u''(z) + \frac{2\nu}{z}u'(z) - \frac{2i}{z^2}u(z) + \frac{2i\nu}{z^3}u(z) - \frac{2i\nu}{z^2}u(z) - \frac{i}{z^2}u(z) - \frac{\nu}{z^2}u(z) \right) = \]
\[ z^\nu e^{-iz} \left( u''(z) + 2\nu u'(z) - 2i\nu u(z) - \frac{2i\nu}{z^2}u(z) + \left( \frac{\nu}{z} \right)^2u(z) - u(z) \right) . \]

With these substitutions Bessel’s equation becomes
\[ u''(z) + 2\nu u'(z) - 2i\nu u(z) - \frac{2i\nu}{z^2}u(z) + \left( \frac{\nu}{z} \right)^2u(z) - u(z) - \frac{\nu}{z}u(z) = 0 = \]
\[ u''(z) + \left( \frac{2\nu + 1}{z} - 2i \right)u'(z) + \left( \frac{\nu}{z^2} - \frac{2i\nu}{z^2} - 1 + 1 + \frac{\nu}{z^2} - \frac{i}{z^2} - \frac{\nu^2}{z^2} \right)u(z) = \]
\[ u''(z) + \left( \frac{2\nu + 1}{z} - 2i \right)u'(z) + \left( \frac{2i\nu}{z^2} - i \right)u(z) \]
multiplying by \( z \) gives
\[ zu''(z) + (2\nu + 1 - 2iz)u'(z) - i(2\nu + 1)u(z) = 0 \]
A solution of this equation is given by the confluent hypergeometric function
\[ \Phi(\nu + 1/2, 2\nu + 1; 2iz) \]
expressing \( b(z) \) in terms of this \( u(z) \) give a solution to Bessel’s equation
\[ b(z) = z^\nu e^{-iz}\Phi(\nu + 1/2, 2\nu + 1; 2iz) \]
or
\[ J_\nu(z) := \frac{1}{\Gamma(\nu + 1)} \left( \frac{z}{2} \right)^\nu e^{-iz}\Phi(\nu + 1/2, 2\nu + 1; 2iz) \]
which differs from \( b(x) \) by a multiplicative constant. \( J_\nu(z) \) is called the Bessel function of the first kind of order \( \nu \).