Lecture 4

Last time

Normal subgroups $G \triangleleft H$

$gHg^{-1} = H \quad gHg^{-1} = H' \in H$

Conjugacy classes - equivalence classes of group elements

consider $S_3$

$geg^{-1} = e$ so $\{e\}$ defines a conjugacy class

$gP_{12}g^{-1}$ it is not hard to show that

(1) this must be an odd permutation - and we get all of them

$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = P_{12}$

etc.

$\{P_{12}, P_{23}, P_{31}\}$ is another class

$\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \}$ is the third class

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

so $S_3$ has 3 disjoint conjugacy classes
Schur's Lemma

(i) Let \( D'(q) \) and \( D^2(q) \) be 2 different irreducible representations of \( \mathcal{G} \) and \( A \) be a linear operator satisfying

\[
D'(q) A = A D^2(q) \quad \forall q \in \mathbb{G}
\]

The \( A = 0 \); if \( D'(q) = D^2(q) \) is irreducible then \( A = \text{constant} \times \text{identity} \).

Proof:
Assume there is a non-zero vector satisfying

\[
A|v\rangle = 0
\]

Then

\[
A D^2(q) |v\rangle = D'(q) A|v\rangle = 0 \quad \forall q \in \mathbb{G}
\]

Since \( D^2(q) \) is irreducible, any vector is a linear combination of \( D^2(q) |v\rangle = 0 \)

\[
A|v\rangle = 0 \quad \forall |v\rangle
\]

If \( A \neq 0 \) and \( |v_1\rangle \neq |v_2\rangle \) then

\[
A|v_1\rangle - A|v_2\rangle = A(|v_1\rangle - |v_2\rangle) \neq 0
\]
must be $A \neq 0$ then $A$ is 1-1.

Next consider - assume there is a $\psi$ satisfying

$$\psi(A) = 0$$

then $\psi(AD^2(q)) = 0 = \psi(D'(q)A)$

this means $\psi(A) = 0$ for all $\psi$ so $A = 0$. Thus if $A \neq 0$ then there is no vector $\psi$ satisfying $\psi(A) = 0$.

this means $A$ is onto.

Thus $A$ is a 1-1 onto linear operator.

This means that it has an inverse since $A$ maps a basis to a basis - both spaces have the same dimension.

It follows that

$$A^{-1}D'(q)A = D^2(q)$$

which contradicts the assumption that $D'(q)$ and $D^2(q)$ are inequivalent.

Next assume

$$D'(q)A = AD'(q) \quad \forall q$$

where $D'(q)$ is irreducible.
A has at least 1 eigenvalue — it might have generalized eigenvectors of rank \( k \)

\[
(A - \lambda I)^k \mathbf{v} = 0 \implies (A - \lambda I) \left\{ (A - \lambda I)^{k-1} \mathbf{v} \right\} = 0
\]

so in this case \( (A - \lambda I)^{k-1} \mathbf{v} \neq 0 \) is an eigenvector.

\[
(A - \lambda I) \mathbf{w} = 0
\]

\[
D_i(g)(A - \lambda I) \mathbf{w} = (A - \lambda I) D_i(g) \mathbf{w} = 0
\]

for all vectors

It follows that \( A = \lambda I \) which completes the proof of Schur's lemma.

**Implication:** Assume \( [H, D(q)] = 0 \) where \( D(q) \) is a unitary representation of a finite group.

Since \( D(q) \) is completely reducible

\[
SD(q) S^\dagger = \bigoplus D_i(q)
\]
$S$ defines a change of basis $\rightarrow \langle nym | D_{m'}(q) | n'm' \rangle$

where $D_{m'}(q)$ is irreducible. In this basis the equation $CHD_{m'}(q) = 0$

$0 = \langle nym | (H \circ D_{m'}(q) - D_{m'}(q) H) | n'm' \rangle =$

$\sum (\langle nym | H | n'm'' \rangle D_{m''}(q) - D_{m''}(q) \langle nym'' | H | n'm' \rangle) = 0 \hspace{1cm} \forall q$

Here for each fixed $nm$, $\langle nym | H | n'm' \rangle$ represents the operator $A$ in Schur's lemma.

The lemma is $\langle nym | H | n'm' \rangle = h_{nm, n'm'}$

In the irreducible basis

$\langle nym | H | n'm' \rangle = h_{nm, n'm'}$

where if $H$ is hermitian, $h_{nm, n'm'}$ is hermitian in each $q$. 


The problem of diagonalizing $H$ can be reduced to diagonalizing $h_{mm'}$ in the irreducible basis.

Next we use Schur's lemma to show that the irreducible representations are an orthogonal set of functions in the regular representation.

Consider $|j m \rangle \langle j' m'|$

where $j m$ and $j' m'$ are fixed $j j'$ label irreducible representations and $m m'$ label vectors in the irreducible subspace.

Define

$$A_{mm'}^{jj'} = \sum_{q \in G} D^j(q) (|j m \rangle \langle j' m'| D^{j'}(q)$$

Consider

$$D^j(q') A_{mm'}^{jj'} =$$

$$\sum_{q \in G} D^j(q') D^j(q) (|j m \rangle \langle j' m'| D^{j'}(q)$$

Let $q'' = q' q$, $q = q'' q'$

$$= \sum_{q} D^j(q'') (|j m \rangle \langle j' m'| D^{j'}(q'' q')$$
summing over $q$ is equivalent to summing over $q''$ for fixed $q'$

\[ D^q(q') A^q_{mm'} = A^{q'}_{mm'} D^q(q') \quad \forall q' \in \mathbb{R} \]

Applying Schur's Lemma $A$ vanishes unless $j = j'$ and if

\[ j = j' \]

\[ A^j_{mm'} = A^{j'}_{mm'} I \]

recall $mm'$ are fixed - $I$ is actually the identity

\[ \langle J\bar{m}' | A^{jj'}_{mm'} | J\bar{m}' \rangle = \]

\[ S^{jj'} S^{\bar{m}'\bar{m}'} A^j_{mm'} \]

we can actually compute the coefficient $A^j_{mm'}$

If we take the trace of the operator

\[ \sum_{mm'} \langle J\bar{m} | A^{jj'}_{mm'} | J\bar{m} \rangle = S^{jj'} A^{j'}_{mm'} \langle J \rangle \]

dimension of $D(C\lambda)$
we can also take the trace of

$$\text{Tr} \left( A_{m'n'}^{ij} \right) = \sum_{m''} D_{mm'}^{j} D_{m'm''}^{i} \left( g' \right) \left( g'' \right)$$

$$= \sum_{m''} D_{mm''}^{j} \left( e \right) = S_{m'm'} S_{j'j} N$$

where $N$ is the order of the group.

Comparing these two expressions for the trace gives

$$S_{j'j} A_{m'm'} N = S_{m'm'} S_{j'j} N$$

$$A_{m'm'} = S_{m'm'} \frac{N}{n_j}$$

This means

$$S_{j'j} S_{m'm'} \frac{N}{n_j} = \sum_{g'} D_{j'}^{j} \langle g' \mid a_{mm'} \rangle \langle a_{mm'} \mid g' \rangle D_{m'm'}^{j}$$

Taking matrix element

$$S_{m'm'} S_{m'm'} S_{j'j} \frac{N}{n_j} = \sum_{g''} D_{m'm'}^{j} \left( g'' \right) D_{m'm'}^{j} \left( g'' \right)$$

In a unitary representation

$$S_{m'm'} S_{m'm'} S_{j'j} \frac{N}{n_j} = \sum_{g'} D_{m'm}^{j} \left( g' \right) D_{m'm'}^{j}$$
This shows that

\[ \sqrt{\frac{n_i}{N}} D_{mm'}^d(g) = \langle g | y \rangle_{mm'} \]

are orthonormal.

Next consider functions \( F(g) \) of group elements.

We define \( \langle FL \rangle \) in the regular representation by

\[ \sum_{g' \in \bar{G}} F(g') \langle g' | = \langle FL \rangle \]

\[ \langle FL | q \rangle = \sum_{g'} F(g') \langle g' | q \rangle = \sum_{g'} F(g') \langle g' | D_{\bar{G}}(g) L_c \rangle \]

\[ = \sum_{g'} F(g') D_{\bar{G}}(g)_{q'q} \]

We can write

\[ \langle q | y \rangle_{mm'} = \sqrt{\frac{n_i}{N}} D_{mm'}^d(g) \]

\[ S_{qq'} = \sum_{mm'} \langle q | y \rangle_{mm'} \langle y | \rangle_{mm'}^{-1} \]

\[ \delta_{yy'} \delta_{mm'} \delta_{mm'}^{-1} = \sum_{q} \langle y | \rangle_{mm'}^{-1} \langle q | y \rangle_{q'q} \]
This can be used to write

\[ \langle F|q \rangle = \sum \langle F|jmm' \rangle \langle jmm'|q \rangle \]

where

\[ \langle jmm'|q \rangle = \sqrt{\frac{n_i}{N}} \, D^{ji}_{mm'}(s) \]

\[ \langle F|jmm \rangle = \sum_q \langle F|q \rangle \sqrt{\frac{n_i}{N}} \, D^{j}_{mm'}(q) \]

so we see that the \( D^{j}_{mm'}(s) \) are an orthogonal basis in functions of \( q \).

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**Characters**

Let \( D(g) \) be a representation of \( G \).

The character of \( D(g) \) is defined by

\[ \chi(g) = \text{Tr}(D(g)) \]

note if \( D'(g) = SD(g)S' \)

\[ \chi'(g) = \text{Tr}(SD(g)S') = \text{Tr}(D(g)S'S) = \text{Tr}(D(g)) = \chi(g) \]
This shows that the characters of equivalent representations are identical.

Next note that if \( q \) and \( q' \) are in the same conjugacy class,
\[
q' = q_1 q_2 q_3^{-1}
\]

Then
\[
\text{Tr} \ D(q') = \text{Tr} \left( D(q_1) D(q_2) D(q_3^{-1}) \right)
\]
\[
= \text{Tr} \left( D(q_1) D(q_2) D(q) \right)
\]
\[
= \text{Tr} \left( D(q) \right)
\]

This shows that the characters of a representation are constant in conjugacy classes.

Consider the orthogonality theorem
\[
\sum_q D_{m\bar{m}}(q) \cdot \frac{\eta_q}{N} D_{m'\bar{m}'}(q) = \delta_{m\bar{m}} \cdot \delta_{m'm'} \cdot \delta_{jj'} \cdot \eta_{j'}
\]

Summing over \( m = \bar{m} \) and \( m' = \bar{m}' \)
\[
\sum_q \left( \sum_{m\bar{m}} D_{m\bar{m}}(q) \right) \cdot \frac{\eta_q}{N} \left( \sum_{m'\bar{m}'} D_{m'\bar{m}'}(q) \right) = \delta_{jj} \sum \delta_{m\bar{m}} \cdot \delta_{m'm'} = \delta_{jj} \cdot N \eta_{j'}
\]

So
\[
\sum_q \chi_{j'}(q) \chi_{j'}(q) = N \delta_{jj'}
\]
This shows that the characters for different irreducible representations are orthogonal vectors. This means that different irreducible representations have different characters.

Example: Consider $\mathbb{Z}_3$

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{bmatrix}
\]

$\chi_e = \chi_\omega = \chi_{\omega^2} = 1$

$\chi_e = 1 \quad \chi_\omega = \omega \quad \chi_{\omega^2} = \omega^2$

$\times$

\[
\begin{pmatrix}
1 \\
\omega \\
\omega^2
\end{pmatrix}
\times
\begin{pmatrix}
1 \\
\omega \\
\omega^2
\end{pmatrix} = 1 + \omega + \omega^2 = 0
\]

\[
\begin{pmatrix}
1 \\
\omega \\
\omega^2
\end{pmatrix}
\times
\begin{pmatrix}
1 \\
\omega^2 \\
\omega
\end{pmatrix} = 1 + \omega + \omega^2 = 0 \quad \omega^2 = \omega
\]

\[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\times
\begin{pmatrix}
1 \\
\omega \\
\omega^2
\end{pmatrix} = 3 \quad \begin{pmatrix}
1 \\
\omega^2 \\
\omega
\end{pmatrix}
\times
\begin{pmatrix}
1 \\
\omega \\
\omega^2
\end{pmatrix} = 1 + 1 + 1 = 3
\]

Next consider functions on the group that are constant on conjugacy classes.

\[
\langle F(1g_1) \rangle = \frac{1}{N} \sum_{g} F(1g_1g) = \text{all the same value}
\]
\[ \frac{1}{N} \sum_{q} \langle \text{Fl}_q \rangle \chi^{*}(q_i) \]

This shows that the characters are a basis for functions that are constant on conjugacy classes.
This means that there are as many conjugacy classes as there are character vectors
\[ \chi^j(E_q, 7) \rightarrow \langle E_1 | E_q, 7 \rangle \]

since
\[ 2 \chi^j(q) \chi^j(5) = NS_{gg'} \]
\[ \sum \chi_i^{[97]} \chi_j^{[97]} \chi_k^{[97]} k \]
\[ \chi^{[97]} \sqrt{\frac{k_s}{N}} \]

are orthonormal functions on conjugacy classes.

Recall last time we proved Cayley's theorem - every finite group is a subgroup of the permutation group.

The conjugacy classes are
\[ \{ e \}, \{ P_{12}, P_{13}, (123) \} \]

\[ e \in \{ e \}, P_{12}, P_{23}, P_{31} \in \{ P_{12}, (123) (123) (231) \} \in \{ (123) \} \]

3 classes, 3 irreducible reps.

\[ N = 6 = 1 + n_1^2 + n_2^2 \quad (1, 1, 2) \]
In this case the classes involve

\((1)(2)(3)\) 1 cycle 1
\((12)(3)\) 2 cycles 3
\((123)\) 3 cycles 1

A general permutation can be decomposed into a product of cycles:

\(c_1 c_2 \cdots c_n\)

\(q^n q'q\) does not change the cycle structure (2 cycles \(\rightarrow\) 2 cycles, 3 cycles \(\rightarrow\) 3 cycles)

Any permutation is a product of cycles.

Counting representations in \(S_n\):

- \(N\) boxes

\[\begin{array}{cccc}
\hline
& & & \\
\hline
\hline
\end{array}\]

- \(N=12\) gives one irrep

- 2 R cycles 1 \(k=1\)

- 12 cycles

- 12 1 cycles