Lecture 5

We used Schur's lemma to show

$$\sum_{g \in G} D_{mm'}^i(g) D_{m'm}^j(g) = \frac{N}{v_i} s_{jj'} s_{mm'} s_{mm'}$$

which shows that the different irreducible representations of $G$ are orthogonal.

If $F(g)$ is a function of group elements (it has $N$ values) then we can write it in the regular representation as:

$$\langle F \mid = \sum_{g \in G} F(g') \langle g' \mid$$

The value of $F(g)$ is computed as:

$$F(g) = \langle F \mid g \rangle = \sum_{g' \in G} F(g') \langle g' \mid g \rangle = \sum_{g' \in G} F(g') s_{g'g}$$

By definition

$$\langle g' \mid g \rangle = D^R_{g'g}(g)$$

where $D^R(g)$ is the regular representation.

$s_{g'g}$ is an $N \times N$ matrix, because the group is finite it can be block diagonalized by a similarity transformation.
This gives

\[ F(q') = \sum_q F(q) \, D_{qe}^R(q') \]

but \( D_{qe}^R(q') \) is completely reducible

\[ F(q') = \sum_q F(q) \, S_{qym} D_{mm'}^d(q') \, S_{ym'e} \]

\[ = \sum_{j mn} \left( \sum_q F(q) \, S_{qym} \, S_{ym'e} \right) D_{mm'}^d(q') \]

\[ = \sum_q F(q) \, C_{mm'}^j(q') \]

\[ \sum_q F(q) \, D_{mm'}^d(q') = 2 \, n^2 \]

both \( D_{mm'}^d(q') \) and \( D_{qe}^R(q') \) are a basis for \( F(q') \)

we get \( N = 2 \, n^2 \)

where the sum is over irreducible representations in the regular representation

where \( D^d \) is an \( n \times n \) irreducible representation.
we can write
\[ \langle F \mid q' \rangle = \sum_{\gamma_{m}'} \left( \sum_{q} \frac{S_{\gamma_{m} q} S_{\gamma_{m} q}^{\dagger} \sqrt{n_{q}}}{\langle \gamma_{m} q \rangle \langle \gamma_{m} q' \rangle} \right) \frac{\sqrt{n_{q}^{\dagger}}}{\langle \gamma_{m} q \rangle \langle \gamma_{m} q' \rangle} \]

The orthogonality relationship gives
\[ \sum_{q} \langle \gamma_{m} q \rangle \langle \gamma_{m} q' \rangle = S_{\gamma_{m} q} S_{\gamma_{m} q}^{\dagger} \]

since both \( |q\rangle \) and \( |\gamma_{m} q\rangle \) are basis vectors in the regular representation.

we also have
\[ S_{\gamma_{m} q} = \sum_{\gamma_{m}'} \langle \gamma_{m} q \rangle \langle \gamma_{m} q' \rangle \]

\[ = \sum_{\gamma_{m}'} \frac{n_{q}^{\dagger}}{n_{q}} D_{\gamma_{m} q}^{\dagger}(q) D_{\gamma_{m} q}^{\dagger}(q') \]

Characters / Conjugacy classes.

If \( D_{\gamma_{m} q}(q) \) is a representation of \( \Gamma \) then the characters of the representation are defined by
\[ \text{Tr} D(q) = \sum_{\gamma_{m}} D_{\gamma_{m} q}(q) \]
The representation does not have to be reducible.

Theorem 1:
equivalent representations have the same characters

\[ D'(\rho) = SD(\rho)S' \]

\[ \chi'(\rho) = \text{Tr}(D'(.)) = \text{Tr}(SD(\rho)S') \]

\[ = \text{Tr}(S'SD(\rho)) = \text{Tr}(D(\rho)) = \chi(\rho) \]

Theorem 2:
the characters of inequivalent irreducible representations are orthogonal

Consider

\[ \sum_{\rho} D_{mm'}^{*}(\rho) D_{\bar{m}\bar{m}'}(\rho) = \frac{N}{n_{j}} S_{jj'}. S_{m\bar{m}}. S_{m'\bar{m}'} \]

set \( m = m' \), \( \bar{m} = \bar{m}' \) and sum

\[ \sum_{\rho} \chi_{j}(\rho) \chi_{j'}(\rho) = \frac{N}{n_{j}} S_{jj'}. \sum_{\bar{m}} S_{m\bar{m}}. S_{m\bar{m}} \]

\[ = \frac{N}{n_{j}} S_{jj'}. n_{j} = S_{jj} N. \]
\[ \sum \chi_i^j(q) \chi_j^i(q) = S_{ij} N \]

which shows that different irreducible representations have different characters.

**Corollary:** The number of different irreducible representations of a group must be less than or equal to the order of the group (we can have at most \( N \) independent vectors in an \( N \)-dimensional space).

**Theorem 3: Characters are constant on conjugacy classes.**

Let \( q_1, q_2 \in G \implies q_1 = g q_2 g^{-1} \) for some \( g \in G \).

\[
D(q_1) = D(q) D(q_2) D(q^{-1})
\]

\[
Tr(D(q_1)) = \chi(q_1) = Tr(D(q) D(q_2) D(q^{-1}))
\]

\[
= Tr(D(q') D(q) D(q_2)) = Tr(D(q'^{-1}) D(q_2))
\]

\[
= Tr(D(q) D(q_2)) = Tr(D(q_2)) = \chi(q_2)
\]

so the characters are actually functions of conjugacy classes.
**Theorem 4** The characters are a basis for functions that are functions of conjugacy classes

\[ \mathcal{F}(q') = \mathcal{F}(qq'q'') \quad \forall q, q' \]

\[ \mathcal{F}(q') = \frac{1}{N} \sum_q \mathcal{F}(qq'q'') = \]

\[ \frac{1}{N} \sum_q \left< F_1 q q' q'' \right> = \]

\[ \frac{1}{2} \sum_q \left< F_{1 jmm'} \right> \sqrt{\frac{N}{n_j}} D_{mm''}^j (q) D_{mm''}^{j*} (q') D_{m'm''m'''}^j (q'') \]

without loss of generality we assume unitary rep

\[ \frac{1}{N} \sum_q \left< F_1 jmm' \right> \sqrt{\frac{N}{n_j}} D_{mm''}^j (q) D_{mm''}^{j*} (q') D_{m'm''m'''}^j (q'') \]

summing over \( q \)

\[ \sum_{n_j} \frac{N}{n_j} \delta_{mm'} \delta_{mm''} \chi_j^j(q) \]

which shows that any function of conjugacy class can be expanded as a linear combination of characters.
Since
\[ \sum_{g} \kappa^2(g) \kappa^2(g') = N S_{gg'} \]
\[ \sum_{g} \kappa^2 \kappa^2 \cdot R_{gg'} = N S_{gg'} \]
\[ \sum_{E_{gg'}} \sqrt{R_{gg'}} \kappa^2 \sqrt{R_{gg'}} \kappa^2 = S_{gg'} \]

where $R_{gg'}$ is the number of elements in the set $G'$, this shows that
\[ \sqrt{R_{gg'}} \kappa^2 \] is an orthonormal basis
in conjugacy classes.

We see $\kappa^2$ are independent in each $G'$
me live on a vector space with dimension =
# conjugacy classes

# irreducible reps \leq # conjugacy classes

since we can expand any function of conjugacy class in terms of $\kappa^2$
# conjugacy classes \leq # irreducible reps.

Taken together we find
**Theorem 6**: The number of conjugacy classes of the group is equal to the number of irreducible representations in the regular representation.

**Example**: $Z_3$

<table>
<thead>
<tr>
<th>Irrs</th>
<th>$e$</th>
<th>$a$</th>
<th>$a^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$e^{2\pi i/3}$</td>
</tr>
<tr>
<td>$1$</td>
<td>$\omega$</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
</tr>
</tbody>
</table>

The values above are the characters:

\[
\begin{pmatrix}
1 & \omega & \omega^2 \\
\omega & \omega^2 & 1 \\
\omega^2 & 1 & \omega \\
\end{pmatrix}
\begin{pmatrix}
1 & \omega & \omega^2 \\
\omega & \omega^2 & 1 \\
\omega^2 & 1 & \omega \\
\end{pmatrix} = 1 + \omega^2 + \omega + 1 = 0
\]

\[
\begin{pmatrix}
1 & \omega & \omega^2 \\
\omega & \omega^2 & 1 \\
\omega^2 & 1 & \omega \\
\end{pmatrix} = 1 + \omega^2 + \omega^2
\]

\[
\{e\} \quad (1,2,3) (1,2,3) (1,2,3) =
\begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{pmatrix} (1,2,3) (1,2,3)
\begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{pmatrix} (1,2,3)
\begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{pmatrix}
\]

Conjugacy classes:

\[
\{e\} \quad \{1,2,3\} \quad \{1,2,3\}
\]

(3 classes - 3 irreducible representations)
Recall

Cayley: every group of order $N$ is a subgroup of the permutation group on $N$ objects.

**Theorem:** elements of the permutation group are products of transpositions.

**Theorem:** each permutation is a product of cycles.

$$(a \rightarrow b \rightarrow c \rightarrow a)(c \rightarrow d \rightarrow e)(\ldots)$$

**Theorem:** permutations with the same cycle structure are in the same conjugacy class.

$$(1 \ 2 \ 3 \ 4) \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$$

$$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 2 & 3 & 4 & 1 \end{pmatrix} \quad \text{(one 4 cycle)}$$

$$(1 \ 2 \ 3 \ 4) \quad 1 \rightarrow 4 \rightarrow 1 \quad 2 \rightarrow 3 \rightarrow 2$$

$$\begin{pmatrix} 4 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad \text{(2 2 cycles)}$$
Each permutation of $N$ objects has a cycle structure.

$N$ boxes:

- $3$ 1-cycles: $(1234)$, $(56)$, $(78)$
- $2$ 2-cycles: $(57)$, $(89)$
- $1$ 4-cycle: $N_{10}$

Each figure gives a different irreducible representation.
Lie groups and Lie Algebras

A Lie group is a group whose elements depend smoothly on some parameters

\[ g = g(\alpha) \in G \]

The group has an identity - it is useful to redefine the parameters so \( \alpha = 0 \) corresponds to the identity

\[ g(0) = e \]

While it is possible to consider the group abstractly - in most physics applications we are interested in representations of the group

\[ D(\alpha) = D(g(\alpha)) \]

\[ D(0) = D(e) = I \]

In a neighborhood of the identity, we can Taylor expand \( D \)

\[ D(\alpha) = I + \alpha \cdot \delta D(\alpha)_{\alpha=0} + \cdots \]
where the dimension of $\xi$ represents the number of group parameters.

It is normal to write this as:

$$D(\alpha) = I + i \alpha \cdot (-i \nabla) \bar{D}(\alpha)_{\alpha=\bar{\alpha}}$$

$$= I + i \sum_a \alpha_a X_a$$

where $X_a = -i \hat{\alpha} \cdot \nabla D(\alpha)_{\alpha=\bar{\alpha}}$

with this definition $X_a$ is a linear operator called a generator of the group.

There are an infinite number of paths that can be taken in moving away from the identity.

We define:

$$D(\Delta \alpha) = I + i \Delta \alpha X_a$$

If we keep taking small steps in a given direction.
If we consider

$$\lim_{n \to \infty} D \left( \frac{a}{n} \right)^n$$

$$\lim_{n \to \infty} \left( 1 + i \frac{a}{n} x_a + \ldots \right)^n = e^{i a x_a} = D(a)$$

where we have used

$$\lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = e^x$$

This is called the exponential parameterization of the group.

For any fixed choice of $\alpha = i \delta$

$$U(\lambda) = e^{i \lambda R \alpha}$$

we have

$$i \lambda R \alpha \cdot i \lambda R \alpha = i (\lambda^2 + \lambda \alpha) R \alpha \cdot R \alpha$$

$$U(\lambda) U(\lambda) = e^{i \lambda R \alpha} e^{i \lambda R \alpha} = e^{i (\lambda + \lambda) R \alpha}$$

$$= U(\lambda + \lambda)$$

since $[R \alpha, R \alpha] = 0$

It is not true that

$$e^{i \lambda R \alpha} e^{i \beta R \alpha} = e^{i (\lambda + \beta) R \alpha}$$

because

$$[R \alpha, R \beta] \neq 0$$
If we consider
\[ i\alpha X_\alpha \ i\beta X_\beta - i\alpha X_\alpha \]
\[ e^e e e e \]
for \( \alpha, \beta \) small it is a group element near the identity so
\[ i\alpha X_\alpha \ i\beta X_\beta - i\alpha X_\alpha \]
\[ e^e e e e e = e \]

Differentiation with respect to \( \alpha, \beta \) setting \( \alpha = \beta = 0 \)
\[
\left( 1 + i\alpha X_\alpha + \frac{1}{2} (i\alpha X_\alpha)^2 \right) \left( 1 + i\beta X_\beta + \frac{1}{2} (i\beta X_\beta)^2 \right) =
\]
\[
1 + i \sum Y_c(\alpha \beta) X_c + \frac{1}{2} \left( i8 Y_c(\alpha \beta) X_c \right)^2 + \cdots
\]
\[ = i^2 \left( X_\alpha X_\beta - X_\beta X_\alpha \right) = \frac{1}{2} \sum \frac{\partial^2 Y_c(\alpha \beta)}{\partial \alpha \partial \beta} X_c \bigg|_{\alpha=\beta=0}
\]
where we have used the fact that \( \left( \frac{\partial}{\partial \beta} \frac{\partial}{\partial \alpha} \right) Y_c(\alpha \beta) = 0 \) since \( \frac{\partial Y_c}{\partial \alpha} = 0 \) at \( \beta = 0 \)

The quantities
\[ S_{abc} = - \frac{\partial^2 Y_c(\alpha \beta)}{\partial \alpha \partial \beta} \bigg|_{\alpha=\beta=0} \]
are called structure constants.
\[ [X_a, X_b] = i \frac{\partial}{\partial x_c} X_c \]

\[ f_{abc} = -\frac{\partial^2 Y_e}{\partial a \partial b} \mid_{a=b=0} \]

In general, since \[ [X_a, X_b] = -[X_b, X_a] \]

\[ f_{abc} = -f_{bac} \]

The relations \( \star \) is called the Lie Algebra of the group.

If the representation \( U(N) \) is unitary, the generators are Hermitian:

\[ [X_a, X_b]^+ = -i f_{abc} X_c \]

\[ [X_b, X_a] = -i f_{bac} X_c \]

\[ [X_a, X_c] = i f_{abc} X_c \]

Subtracting

\[ 0 = i (f_{abc} - f_{bac}) X_c = 0 \]

since \( X_c \neq 0 \) this means in a unitary representation the structure constants must be real.
For commutators in general

\[ [X_a [X_b X_c] + [X_b [X_c X_a] + [X_c [X_a X_b]] = 0 \]

It can be expressed as

\[ [X_a [X_b X_c] = \]

\[ [X_a X_b] X_c + [X_b [X_a X_c]] \]

which looks like the chain rule for derivatives.

Consider

\[ [X_a [X_b X_c] = \]

\[ [X_a f_{bcd} X_b] = \]

\[ f_{bcd} [X_a X_c] = (ii) f_{bcd} f_{ade} X_e \]

\[ -f_{bcd} f_{ade} X_e \]

Adding cyclic permutation,

\[ -f_{bcd} f_{ade} + f_{cad} f_{ade} + f_{abc} f_{cde} X_e = 0 \]

Since the \( X_e \) are independent and non-0,

\[ f_{bcd} f_{ade} + f_{cad} f_{ade} + f_{abc} f_{cde} = 0 \]
If we define matrices

\[ [T_a]_{bc} = -i \delta_{abc} \]

then the equation can be expressed as

\[ [T_b]_{cd} [T_a]_{de} \]
\[ [T_c]_{ad} [T_b]_{de} \]
\[ [T_a]_{bd} [T_c]_{de} = 0 \]

using the antisymmetry,

\[ [T_b]_{cd} [T_a]_{de} - [T_a]_{cd} [T_b]_{de} = - ( -i \delta_{aba} ) (-f_{dé} )_{ce} \]

\[ T_b T_a - T_a T_b = i f_{bad} T_a \]

which shows that the structure constants generate a representation of the Lie Algebra with dimension \# of independent generators.

This is called the adjoint representation.

The generators are operators, but they also represent independent vectors.
we can define an inner product as follows.

Consider \( \text{Tr}(T_a T_b) \)

Clearly \( \text{Tr}(T_a T_b) = \text{Tr}(T_b T_a) = \text{Tr}(-T_a^* T_b^*) \)
\[ = \text{Tr}(T_b T_a)^* = \text{Tr}(T_a T_b)^* \]

This is in general a real symmetric matrix (for \( X = X^* \rightarrow \text{real} \))

Consider
\[ X_a \rightarrow X_a' = M_{ab} X_b \]

Then
\[ [X_a', X_b'] = M_{ac} M_{bd} [X_c X_d] \]
\[ = i M_{ac} M_{bd} f_{cde} X_e \]
\[ = i M_{ac} M_{bd} f_{cde} M_{ef} X_f' \]

under this transformation the structure constants change
\[ f'_{abc} = M_{ac} M_{bd} M_{ef} f_{cde} \]
Under this transformation the adjoint representation becomes

\[ [T^\prime_{abc}] = -i f^\prime_{abc} = \]
\[ = -i M_{ad} M_{bc} M_{dc} f_{def} \]
\[ = M_{ad} (M T d M^{-1})_{bc} \]

In matrix notation:

\[ [T^\prime_a] = M_{ad} (M T d M^{-1}) \]

If we consider

\[ \text{Tr} (T^\prime_a T^\prime_b) = \]
\[ = M_{ac} M_{bd} \text{Tr} (M T c M^{-1} M T d M^{-1}) \]
\[ = M_{ac} M_{bd} \text{Tr} (T_c T_d) \]
\[ = M_{ac} \text{Tr} (T_c T_d) M_{bd} \]

Since \( \text{Tr} (T_c T_d) \) is a real hermitian matrix it can be diagonalized using a real orthogonal matrix.
\[
M|\psi\rangle = \lambda|\psi\rangle \\
M^*|\psi^*\rangle = \lambda^*|\psi^*\rangle \\
M|\psi^*\rangle = \lambda|\psi^*\rangle \quad (\text{since } M \text{ are real}) \\
|\psi^\prime\rangle = |\psi\rangle + |\psi^*\rangle; \quad |\psi^\prime\rangle = i(|\psi^*\rangle - |\psi\rangle)
\]

are both real — so the eigenvectors can be chosen to be real. — since the columns of \( M \) are eigenvectors:
\[ M = M^* \quad M^+ = M^T \quad M M^T = I \]

We choose \( M \) so it diagonalizes
\[ \text{Tr}(X_a X_b^\prime) = \kappa_a \kappa_b \]

In general the \( \kappa_a \) could be positive, negative, or 0.

Multiply
\[
M \rightarrow \begin{pmatrix} \frac{1}{\sqrt{\kappa_a}} \\ \frac{1}{\sqrt{\kappa_b}} \end{pmatrix} M
\]

can make all of the \( \kappa_a \), \( \pm \pm \pm \pm \)

(we can't change the sign because \( M \) appears twice)
In what follows we consider groups where all of the $k_a > 0$ then we can choose $M M'$ so

$$\text{Tr} (T_a T_{b'}) = c \delta_{ab}$$

with this choice

$$\text{Tr}([T_a T_{b'} T_c]) =$$

$$i \lambda f'_{abc}$$

so

$$i \lambda f'_{abc} = \text{Tr}([T_a T_{b'} T_c]) =$$

$$\text{Tr}(T_a T_{b'} T_c - T_{b'} T_a T_c)$$

$$\text{Tr}(T_a T_{b'} T_c - T_a T_{b'} T_c)$$

$$\text{Tr}(T_a T_{b'} T_c - T_c T_{b'} T_a)$$

$$\text{Tr}([T_a T_{b'} T_c]) = i \lambda f'_{bca}$$

so

$$f_{abc} = -f_{bca} = f_{bca} = -f_{cba} =$$

$$= f_{cab} = -f_{acb}.$$

In this basis

(i) The $f_{abc}$ are completely

antisymmetric
3. $T_a$ are pure imaginary and are antisymmetric as matrices (so they are still Hermitian)
3. The adjoint representation is unitary

An invariant subalgebra is a set of generators $Y_a$ such that

$$[Y_a, X_b] = i f_{abc} Y_c$$

$Y_a \in$ sub algebra $X_a \in$ algebra.

Invariant subalgebras are related to normal subgroups:

$$h = e^{-iy} \quad q = e^{ix}$$

$$ghg^{-1} = e^{ix} he^{-iy} =$$

$$e^{i e^{i y} e^{-x} (1 + i [x,y] + \frac{1}{2} [x,[x,y]] + \cdots)}$$

Each commutator is in the subalgebra $i \in \mathcal{C}$. $a, b, c$,

which means that $e^{i e^{i y} e^{-x} (1 + \cdots)}$ are elements of an invariant subgroup.
An algebra which has no trivial invariant subalgebra generates a simple group.

The adjoint representation of a simple algebra is irreducible.

Proof: Assume that there is a non-trivial invariant subspace in the adjoint representation:

\[ T_1 \ldots T_n \in S \]
\[ T_{n+1} T_n \in S_1 \]

\[ [T_a]_{nm} = 0 \text{ nes } m \in S_1 = -i f_{amn} \]
\[ f_{amn} = 0 \text{ in all nes } m \in S_1 \]

By the complete antisymmetry, none of the elements involve either all nes in all m in S_1.

There are 2 invariant subalgebras.

The last class of algebras are semisimple algebras - these have no abelian invariant subalgebras.