Lecture 6

Lie groups and Lie algebras

A Lie group whose elements depend smoothly on some parameters $\alpha$

$g(\alpha) \in G$

we are mostly interested in representations of Lie groups

$D(g(\alpha)) := D(\alpha)$

we are free to change variables in the parameter space. It is useful to choose $\alpha = 0$ to represent the identity

$g(0) = e \quad D(0) = I$

For small $\alpha$, $D(\alpha)$ is close to the identity. For small $\alpha$, the Taylor expansion of $D(\alpha)$ has the form

$D(\alpha) = I + \alpha \cdot \frac{\partial}{\partial \alpha} D(\alpha)_{\alpha=0} + o(\alpha^2)$

we define the operam

$\mathbf{2} = -i \frac{\partial}{\partial \alpha} D(\alpha)_{\alpha=0}$
The reason for the $i$ is if $D(g(x))$ is a unitary representation then $E$ is Hermitian.

Since these are group elements the representation property means $D(\alpha)D(\alpha)^\dagger = D(\alpha)$ is a group element.

Consider $D(g(n))$ - this has an expansion for fixed $x$

$$(1+i \frac{\partial}{\partial x}E+\cdots)(1+i \frac{\partial}{\partial x}E+\cdots) = (1+i \frac{\partial}{\partial x}E)$$

To calculate $D(\bar{g})$ in infinite $\alpha$ consider the limit

$$\lim_{n \to \infty} \left(1+i \frac{\partial}{\partial x}E+\cdots\right) = i \bar{E} \cdot \bar{\alpha}$$

Since this converges this is the exponential parameterization of the group near the identity

In general $\bar{\alpha} \cdot \bar{E}$ and $\bar{\beta} \cdot \bar{E}$ do not commute, however

$$i \bar{\alpha} \cdot \bar{E} \quad i \bar{\beta} \cdot \bar{E} = -i \bar{\alpha} \cdot \bar{E}$$

is a group element - it is near the identity in small $\bar{\alpha}$ and $\bar{\beta}$.
\[ E = \sum \alpha_a L_a \]

Consider the components
\[ i \lambda L_a L_b - i \lambda L_a L_c + \sum_{\lambda \neq \mu} \frac{2}{\lambda + \mu} \gamma_{\lambda}(\lambda \mu) L_c \]
for \( \lambda, \mu \) near \( 0 \). \( \gamma_{\lambda}(\lambda \mu = 0) = 0 \), \( \gamma_{\lambda}(0,0) = 0 \).

Calculating \( \frac{\partial^2}{\partial \lambda \partial \mu} |_{\lambda = \mu = 0} \) gives

\[ \left( L_a L_b - L_b L_a \right) = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \]

\[ i \sum_{\lambda = 0} \frac{\partial \gamma_{\lambda}}{\partial \lambda} L_c + i \sum_{\lambda = 0} \sum_{\mu = 0} \left( \frac{\partial^2 \gamma_{\lambda}}{\partial \lambda \partial \mu} \gamma_{\lambda} + \gamma_{\lambda} \frac{\partial^2 \gamma_{\lambda}}{\partial \mu^2} + \frac{\partial \gamma_{\lambda}}{\partial \lambda} \frac{\partial \gamma_{\lambda}}{\partial \mu} + \frac{\partial \gamma_{\lambda}}{\partial \lambda} \frac{\partial^2 \gamma_{\lambda}}{\partial \mu^2} \right) L_c L_c \]

This gives

\[ L_a L_b - L_b L_a = -i \sum_{\lambda = 0} \frac{\partial^2 \gamma_{\lambda}}{\partial \lambda \partial \mu} L_c \]

We define
\[ \gamma_{\lambda \mu} = - \frac{\partial^2 \gamma_{\lambda}}{\partial \lambda \partial \mu} |_{\lambda = \mu = 0} \]

These numbers are called structure constants.
The linear operators $L_a$ are elements of the Lie algebra of the Lie group.

We have just shown that they satisfy the commutation relations

$$\{L_a, L_b\} = i \sum f_{abc} L_c$$

**Remarks**

1. Since $\{L_a, L_b\} = -\{L_b, L_a\}$ then

   $$f_{abc} = -f_{bac}$$

2. If $\tilde{L} = L^t$

   $$\{L_a, L_b\}^t = \{L_b^t, L_a^t\} = \{L_b, L_a\} = i \sum f_{bac} L_c^t$$
   $$= -i \sum f_{bac} L_c$$
   $$= -i \sum f_{bac} L_c$$

   Comparing both sides

   $$f_{abc} = f_{bac}^* \quad (\text{Hermitian } L)$$

3. $$\{L_a, \{L_b, L_c\}\} =$$

   $$i \sum_{d} \{L_a, f_{bcd} L_d\} =$$

   $$i^2 f_{bcd} f_{ade} L_e$$

   by the Jacobi identity, summing over cyclic permutations gives $0$
\[-f_{bcda} f_{ade} L + \]
\[-f_{cad} f_{bde} L + \]
\[-f_{ab} f_{cde} L = 0\]

Since each of the \( L \) are independent linear operators

\[f_{bcda} f_{ade} + f_{cad} f_{bde} + f_{ab} f_{cde} = 0\]

Next we define the matrices

\[(X^a)_{bc} = -i f_{abc} \quad f_{abc} = i X^c_{bc}\]

Using this in the above expression gives

\[
\begin{align*}
(i)^2 (X^b)_{cd} X^a_{de} - (i)^2 X^a_{cd} X^b_{de} - f_{abcd} X^d_{ce} \quad \text{anti-symmetric} \\
- X^b X^a + X^a X^b &= i f_{abcd} X^d \quad \text{in}
\end{align*}
\]

As matrices

\[
[X^a, X^b] = i f_{abcd} X^d
\]

so the structure constants themselves form a representation of the Lie algebra.
This is called the adjoint representation. The dimension of the adjoint representation is equal to the number of independent group parameters \( = \# \text{ of independent generators} \).

Note since \( f_{abc} \) is real, for a unitary representation of the group, the \((X^a)_{bc}\) are all pure imaginary.

Since the \( X^a \) are also vectors, it is useful to construct an inner product.

Consider

\[
\text{Tr}(X^a X^b) = \text{Tr}(X^b X^a) = \\
\text{Tr}(-X^a (-X^b)^\dagger) = \text{Tr}(X^a X^b)^* 
\]

This is a real symmetric matrix, it can be diagonalized by a real orthogonal matrix

\[
M |v\rangle = \lambda |v\rangle
\]

\[
M^* |v\rangle^* = \lambda |v\rangle^* \quad (M = M^T)
\]

\[
M |v\rangle^* = \lambda |v\rangle^* \quad (M = M^*)
\]

Then \( |v\rangle |v\rangle^* \) can be replaced by

\[
|v\rangle + i |v\rangle^* \quad \text{or} \quad |v\rangle - i |v\rangle^*
\]
which are both real.

Consider a general linear transformation on the $X^a$

$$\left[ X^a, X^b \right] = i \delta_{ac} X^c$$

$$X'^a = M^{ap} X^p$$

$$\left[ X'^a, X'^b \right] = \sum M^{ac} M^{bd} \left[ X^c, X^d \right] = \\ \sum M^{ac} M^{bd} i f_{cde} M^{-1}_{ef} X^f$$

$$\left[ X'^b, X'^c \right] = i \Sigma \left( M^{ac} M^{bd} f_{cde} M^{-1}_{ef} \right) X^f$$

under this change of basis the structure constants change,

$$f_{abc} \rightarrow f'_{abc} = M^{ac} M^{bd} f_{cde} M^{-1}_{ef}$$

The adjoint representation also transform,

$$\left( X'^a \right)_{bc} = - i f'_{abc} = M^{ac} M^{bd} \left( : X^c_{de} \right) M^{-1}_{ef}$$

$$\left( X' \right)^a = \frac{M^{ac} \left( M X M' \right)}{\text{comp. matrix}}$$
If we relate the traces

$$\text{Tr} \left( X^a X^b \right) = M^{ac} M^{bd} \text{Tr} (M X^c M^d, M X^d M^c)$$

$$= M^{ac} M^{bd} \text{Tr} (X^c X^d)$$

$$= M^{ac} \text{Tr} (X^c X^d) (M^T)^{db}$$

since the original matrix was real symmetric we can choose an orthogonal M ($MM^T = I$) that diagonalizes $\text{Tr} (X^c X^d)$

$$\text{Tr} (X^a X^b) = \lambda^a \delta_{ab}$$

In general the eigenvalues can be positive, negative or 0.

multiplied by

$$N \begin{pmatrix} a_1 & 0 \\ 0 & a_n \end{pmatrix} \text{ with } a_n = \begin{cases} \frac{1}{\sqrt{a_1}} & \text{if } \lambda^a > 0 \\ 1 & \text{if } \lambda^a = 0 \\ \frac{1}{\sqrt{|\lambda^a|}} & \text{if } \lambda^a < 0 \end{cases}$$

$$\text{Tr} (X^a X^b) \rightarrow \eta_a \delta_{ab} \quad \eta_a = 0, 1, -1$$

this simply rescales the length of $X$
A transformation $X \rightarrow X' = MX$ cannot change the sign of the eigenvalue of $\text{Tr}(X'X'')$.

The algebras where all of the $\lambda_a$ are strictly $> 0$ are called compact Lie algebras.

Consider the compact case:

\[
\text{Tr} (X_a X_b) = C S_{ab} \\
\text{Tr} (X_a X_b X_c) = \\
2\text{Tr} (i g_{abc} X_a X_b X_c) = \\
i 2 \delta_{abcd} S_{ac} C = \\
\text{Tr} (X_a X_b X_c) = i f_{abc} = \\
\text{Tr} (X_a X_b X_c - X_b X_a X_c) = \\
\text{Tr} (X_b X_c X_a - X_c X_b X_a) = \\
\text{Tr} (X_a X_b X_c) = i f_{abc} \\
f_{abc} = f_{bca} = f_{cab} = \\
f_{bac} = -f_{cb} = -f_{acb}$
The means that in compact groups it is always possible to transform the generators of a compact group so the structure constants are completely antisymmetric.

In the adjoint representation $T_a$ are (1) pure imaginary and (2) antisymmetric

$$X^+ = -X^T = X$$

so they are Hermitian - the adjoint representation of the group constructed by $e^{i\Xi}$ is unitary.

Invariant subalgebras - invariant subgroups

$\mathfrak{y}_a$ is a sub algebra of $\{X_a\}$ if

$$[Y_a, X_b] = -i \delta_{abc} Y_c \quad \forall X_a \in \{X_a\}$$

implies

$$q^{-1} \mathfrak{y}_a q = e^i \beta \cdot (i\alpha x - i\omega y)$$

$$e^{i\beta \cdot (i\alpha x + i\omega y)}$$
\[
e^{i\beta_y} + i \left[ Q, \chi, \beta_y \right] + \frac{j^2}{2i} \left[ Q,\chi, Q,\chi, \beta_y \right]
\]

we see each term gives a generator of the \( Y \) subgroup which means
\[q h q' \in H\]
\(H\) is a normal subgroup \( \subset G\).

example \( SU(2) \) 2x2 unitary matrices

2x2 unitary matrices
with \( \det M = 1 \)
\[U^* = U^t \quad \text{and} \quad \det U = 1\]

\[U = e^{iH} \quad H = H^* \quad \Rightarrow U \text{ is unitary}\]

\[H = H^t = \begin{pmatrix} h_{00} + h_{33} & h_{12} + i h_{13} \\ h_{12} - i h_{13} & h_{00} - h_{33} \end{pmatrix} = h_{00} I + \vec{h} \cdot \vec{\sigma}\]

\[\text{Tr} (I) = 2 \quad \text{Tr} (\vec{\sigma}) = 0\]

\[\det U = \det e^{iH} = e^{i \text{Tr} H} = e^{2i \text{Re}(H)} = 1\]

\[h_{00} = 0; \quad \Rightarrow \quad U = e^{i \vec{h} \cdot \vec{\sigma}}\]
In this case we see \( n_1, n_2, n_3 \) are the group parameters. If we choose

\[
U(n) = e^{i n \cdot \frac{\vec{G}}{2}}
\]

then we can calculate the structure constant

\[
U(n) \rightarrow (1 + i n \cdot \frac{G}{2} + \cdots)
\]

\[
\left[ \frac{G_i}{2}, \frac{G_j}{2} \right] = \frac{i}{4} \left( S_{ij} + i \epsilon_{ijk} G_k - S_{ij} - i \epsilon_{ijk} G_k \right)
\]

\[
= \frac{i}{2} \epsilon_{ijk} \frac{G_k}{2}
\]

In this case the the structure constants are \( \epsilon_{ijk} \)

\[
(J^i)_{jk} = -i \epsilon_{ijk}
\]

These are 3x3 matrices

\[
J^1 = -i \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}
\]

\[
J^2 = -i \begin{pmatrix}
0 & 0 & -1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

\[
J^3 = -i \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
\[
\begin{align*}
\hat{J}_1^x &= (0,0,0) \hat{x} \\
\hat{e} &= e \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \end{pmatrix} \\
\hat{J}_2^x &= (0,0,-1) \hat{x} \\
\hat{e} &= \begin{pmatrix} \cos x & 0 & -\sin x \\ 0 & 1 & 0 \\ \sin x & 0 & \cos x \end{pmatrix}
\end{align*}
\]

**Basis states - construct commuting Hermitian operators**

\[
\left[ \hat{J}_1^2, \hat{J}_2^2, \hat{J}_3^2, \hat{J}_1^z \right] = 0,
\]

\[
\frac{\hbar}{\kappa} \left[ \hat{J}_1^3, \hat{J}_1^z \right] = \frac{\hbar}{\kappa} \left( \kappa \hat{J}_1 \left[ \hat{J}_1 \hat{J}_1^2 \right] + \left[ \hat{J}_1 \hat{J}_1^2 \right] \hat{J}_1 \right)
\]

\[
i \epsilon_{R12} \left( \hat{J}_1 \hat{J}_2 + \hat{J}_2 \hat{J}_1 \right) = 0
\]

**Diagonalize** \(\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2, \hat{J}_3\)

**Construct operators that change \(\hat{J}_1^z\)**

\[
\left[ \hat{J}_3, \hat{J}_1 \mp i \hat{J}_2 \right] = i \hat{J}_2 \pm 1 (-i \hat{J}_1)
\]

\[
= \pm (i \hat{J}_1 + i \hat{J}_2)
\]

(lie the other 2 operators)
Let $|\lambda, \eta\rangle$ be simultaneous eigenstates of $J^2$ and $J_3$.

\[ J^2 |\lambda, \eta\rangle = \lambda |\lambda, \eta\rangle \]
\[ J_3 |\lambda, \eta\rangle = \eta |\lambda, \eta\rangle \]
\[ J_3 J_\pm |\lambda, \eta\rangle = J_\pm J_3 |\lambda, \eta\rangle \pm J_3 |\lambda, \eta\rangle \]
\[ = (\eta \pm 1) |J_\pm |\lambda, \eta\rangle \]

Also note

\[ J_+ J_- = (J_1 J_2 - i J_3)(J_1 J_2 + i J_3) \]
\[ = J_1^2 J_2^2 + J_3^2 \pm i [J_1 J_2, J_3] \]
\[ = J_1^2 J_2^2 + J_3^2 \pm i (i J_3^2) - J_3^2 \]
\[ = J_1^2 - J_3^2 \pm J_3 \]
\[ = J_1^2 - J_3 (J_3^2 + 1) \]

\[ J_+ |\lambda, \eta\rangle = C |\lambda, \eta \pm 1\rangle \]

\[ \langle \lambda \eta | J_+ J_- | \lambda \eta \rangle = C^2 = \langle \lambda \eta | J_1^2 J_2^2 - J_3 (J_3^2 + 1) | \lambda \eta \rangle \]
\[ = \lambda - \eta (\eta \pm 1) \geq 0 \]

This eventually becomes negative for fixed $\lambda = \eta$

\[ \eta^* (\eta^* \pm 1) = \lambda \quad \text{= extremal value} \]

Solutions: $\lambda = \eta_{\max} (\eta_{\max} + 1)$
\[ = (-\eta_{\max})(-\eta_{\max} - 1) \]
\[ = \eta_{\max} (\eta_{\max}^* + 1) \]
Then we determine

\[ \lambda \sim n_{\text{max}} \]

\[ J^2 | n_{\text{max}} n \rangle = n_{\text{max}} (n_{\text{max}} + 1) | n_{\text{max}}, n \rangle \]

\[ J_3 | n_{\text{max}} n \rangle = n | n_{\text{max}} n \rangle \]

\[ -n_{\text{max}} + 1 \leq n \leq n_{\text{max}} - 1 \leq n_{\text{max}} \]

Since there are an integer \# of steps \[ 2n_{\text{max}} = \text{integer} \]

This gives \[ n_{\text{max}} = \frac{n}{2} \]

The adjoint representation

\[ J_1^2 + J_2^2 + J_3^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2I \]

These steps get generalized for other compact groups.