Lecture 8

$SU(2)$

$U^\dagger = U^{-1} \Rightarrow U = e^{iM}$, $M = M^\dagger$

$\det U = e^{i \text{Tr}(M)}$

$M = M^\dagger \Rightarrow M = m_0 I + \bar{m} \cdot \vec{\sigma}$

$\vec{\sigma} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$\sigma_i = \sigma_i^\dagger$, $\text{Tr}(\sigma_i) = 0$, $\sigma_i \sigma_j = \delta_{ij} I + i \varepsilon_{ijk} \sigma_k$

$\text{Tr}(M) = 2m_0 \Rightarrow 2m_0 = 0, 2\pi$, $m_0 = 0, \pi$

This means:

$U = \pm e^{i \vec{\sigma} \cdot \vec{m}}$

It is customary to choose the parameters so

$U = e^{i \frac{\vec{\sigma} \cdot \vec{m}}{2}} = \cos \left( \frac{|m|}{2} \right) + i \hat{m} \cdot \vec{\sigma} \sin \left( \frac{|m|}{2} \right)$

when $|m| = 2\pi$, $U = -I$, which means that

$U = e^{i \frac{\vec{\sigma} \cdot \vec{m}}{2}}$ includes $U = e^{i \frac{\vec{\sigma} \cdot \vec{m}}{2}}$

generators

$L_i = -i \frac{\partial}{\partial m_i} \left( e^{i \frac{\vec{\sigma} \cdot \vec{m}}{2}} \right) \Big|_{m = 0} = \frac{\sigma_i}{2}$

$L_i = \frac{\sigma_i}{2}$
structure constants:

\[ [L_i, L_j] = \left[ \frac{\sigma_i}{2}, \frac{\sigma_j}{2} \right] = \frac{1}{4} \left( \sigma_i \sigma_j - \sigma_j \sigma_i \right) \]

\[ = \frac{1}{4} \left( \delta_{ij} I + i \varepsilon_{ijk} \sigma_k \right) = \frac{1}{2} \varepsilon_{ijk} \sigma^k \]

\[ \frac{\varepsilon_{ijk}}{2} = \frac{1}{2} \varepsilon_{ijk} \sigma^k \]

with this parametrization the structure constants are already antisymmetric adjoint representation

\[ X^i_{jk} = -i \varepsilon^{ijk} \]

there are 3x3 matrices \( 3 = \) number of group parameters

\[ X^1 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \]

\[ X^2 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \]

\[ X^3 = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ i & 0 & 0 \end{pmatrix} \]
\[ X^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & -i \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ \text{Tr} \left( X^2 X^2 \right) = 2 \]

\[ (X^2)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & -i \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ \text{Tr} \left( X^2 X^2 \right) = 2 \]

\[ (X^3)^2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ \text{Tr} \left( X^3 X^3 \right) = 2 \]

\[ \text{Tr} \left( X^i X^j \right) = 2 \delta_{ij} \]

**Multiplication law**

\[ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |x_1\rangle \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |x_2\rangle \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = |x_3\rangle \]

\[ X^j |x_k\rangle = \sum_m |x^m\rangle <x^m|X^j|x_k\rangle = \sum_m |x^m\rangle X^j_{mk} = \sum_m |x^m\rangle (-i \epsilon_{jkm}) \]

\[ = \sum_i \epsilon_{ijk} m |x^m\rangle = |[X^j X^k]\rangle \]
cuntan subalgebra =

maximal set of commuting generators - in this case there is only 1. We take
$X^3$ - dim of cuntan subalgebra = rank

In the adjoint representation

$X^3 | X^3 \rangle = \{ [X^3 X^3 \rangle = 0$

\[
\begin{pmatrix}
0 & -i & 0 \\
0 & 0 & 0 \\
i & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} = 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

weights

$X^3 | \mu \rangle = \mu | \mu \rangle$

\[
\text{det} \left( \begin{pmatrix}
0 & -i & 0 \\
0 & 0 & 0 \\
i & 0 & 0
\end{pmatrix} - \begin{pmatrix}
\mu & 0 & 0 \\
0 & \mu & 0 \\
0 & 0 & \mu
\end{pmatrix} \right)
\]

\[
\text{det} \begin{pmatrix}
-\mu & -i & 0 \\
-\mu & 0 & 0 \\
0 & 0 & -\mu
\end{pmatrix} = -\mu^3 - (i)(-i)(-\mu)
\]

$= -\mu^3 + \mu = -\mu (\mu^2 - 1)$

$= \mu = 0, \mu = \pm 1$

the eigenvalues $\mu_i$ of $X^3$ are called weights. The eigenvectors are called weight vectors.
The eigenvectors are

\[ |u=\pm 1\rangle = C ( |x^1\rangle + i |x^2\rangle ) \]

\[ |u=-\pm 1\rangle = C ( |x^1\rangle - i |x^2\rangle ) \]

\[ X^3 |u=\pm 1\rangle = C \left( X^3 |x^1\rangle \pm i X^3 |x^2\rangle \right) \]

\[ = C \left( X^3 |x^1\rangle \pm i [X^3 |x^2]\rangle \right) \]

\[ = \pm 1 |u=\pm 1\rangle \]

\[ = \pm 1 C \left( |x^1\rangle \pm i |x^2\rangle \right) \]

This means

\[ [X^3, X^{1\pm i}X^2] = \pm 1 (X^{1\pm i}X^2) \]

We can check this with the matrix representation:

\[ X^{1\pm i}X^2 = \begin{pmatrix} 0 & 0 & \mp i \\ 0 & 0 & -i \\ \pm i & -i & 0 \end{pmatrix} \]
\[
\begin{pmatrix}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & i
\end{pmatrix}
\begin{pmatrix}
0 & 0 & -i \\
0 & 0 & -i \\
-i & i & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & -i \\
0 & 0 & -i \\
-i & i & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-1 & i & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
X_1^3, X_1^{\pm} iX_1^2 \end{pmatrix} = \begin{pmatrix}
0 & 0 & -1 \\
0 & 0 & -i \\
1 & i & 0
\end{pmatrix} = \pm \begin{pmatrix}
0 & 0 & -1 \\
0 & 0 & -i \\
1 & i & 0
\end{pmatrix} = X_1^{\pm} iX_1^2
\]

we identify \( \Lambda = 1 \) as the highest weight.

In general \( X_1^\pm = X_1^{\pm} iX_1^2 \)

\[
\begin{pmatrix}
X_1^3, X_1^{\pm} \end{pmatrix} = \mu X_1^{\pm}
\]

taking adjoint

\[
\begin{pmatrix}
X_1^{\pm \dagger}, X_1^3 \end{pmatrix} = \mu(X_1^\dagger)^\dagger
\]

\[
\begin{pmatrix}
X_1, x_1^{\pm \dagger} \end{pmatrix} = -\mu X_1^{\pm \dagger}
\]

\[
(X_1^\dagger)^\dagger = (X_1)^\dagger
\]
\[ X^3 X^\pm 1 u > = (X^\pm X^3 \pm X^\pm) 1 u > = 0 \]
\[ = (u \pm 1) 1 X^\pm 1 u > \]

This means
\[ X^\pm 1 u > = (1 \pm 1) u > \]

Since
\[ X^\pm 1 X^T > = 1 [ X^\pm X^T ] > \]
\[ X^3 X^\pm 1 X^T > = (X^\pm X^3 \pm X^\pm) 1 X^T > \]
\[ = X^\pm [X^3 X^\pm ] > \mp X^\pm 1 X^T > \]
\[ = X^\pm 1 \neq X^T > \pm X^\pm 1 X^T > \]
\[ = (\mp \pm 1) X^\pm 1 X^T > = 0 \]

\[ [ X^\pm X^T ] > \text{ is in the center of the algebra} \]

In this case
\[ \langle X^3 | X^\pm | X^T > = \]
\[ \langle X^3 | [X^\pm X^T ] > = \]
\[ \frac{1}{2} \text{Tr} ( X^3 X^\pm X^T - X^3 X^T X^\pm ) = \]
\[ \frac{1}{2} \text{Tr} ( X^\pm [X^T X^3 ] ) = \]
\[-\frac{1}{2} \text{Tr} (X^\pm [X^3, X^\mp])\]
\[= \frac{1}{2} (\pm 1) \text{Tr} (X^\pm, X^\pm)\]
\[= \frac{1}{2} (\pm 1) \text{Tr} (X^\pm, X^\mp)\]
\[= \pm 1 \cdot \frac{1}{2} \cdot 2 = \pm 1\]

This gives
\[\left[ X^\pm, X^\mp \right] = \pm X^3 \quad (u, \tilde{u}, \tilde{x}^3)\]

This is because \([X^\pm, X^\mp] \] is in the center subalgebra so it is \(a \times X^3\); \(\pm 1\) is the coefficient.

These are familiar relations in \(SU(2)\)
\[X^3 \rightarrow S^3\]
\[X^\pm = S^1 \pm is^2\]
\[\left[ S^\pm, S^3 \right] = \pm S^3\]

Note that these concepts apply to compact Lie groups (see chapter 6-8)
Infinite dimensional vector spaces

example: complex valued continuous functions on the interval $x \in [0, b]$

is $f(x)$ and $g(x)$ are continuous and $c$ is complex

(1) $f(x) + g(x)$ is continuous on $[0, b]$

(2) $cf(x)$ is continuous on $[0, b]$

(3) $f(x) = 0$ is continuous

$g(x) + 0 = g(x)$

In every continuous $g(x)$

$g(x) + (-g(x)) = 0$

These satisfy all of conditions for the space of continuous functions to be a vector space.

It is useful to think of the value of $f$ at $x$ as a component of a vector $f$.

Inner products

Let $\omega(x) > 0$ for $x \in [0, b]$. 
\[ \langle f | g \rangle = \int_a^b \omega(x) f^*(x) g(x) \, dx \]

with this definition

(1) \[ \langle f | f \rangle = \int_a^b |f(x)|^2 \omega(x) \, dx \geq 0 \]

(2) if \[ \langle f | f \rangle = 0 \] this means

\[ \int_a^b |f(x)|^2 \omega(x) \, dx = 0 \]

since \( f(x) \) is continuous this require \( f(x) = 0 \) for all \( x \)

proof (if \( f(x) > 0 \) there is a an interval of width \( \varepsilon \) where \( f(x) > \frac{1}{2} f(x_i) \), \( x \in [x_i - \varepsilon, x_i + \varepsilon] \).

\( \omega(x) \) has a minimum value on this interval \( \omega' \)

\[ \int_a^b |f| \omega(x) \, dx \geq 2 \varepsilon \cdot \frac{1}{2} f(x_i) \omega_x > 0 \]

giving a contradicitm

(3) \[ \langle f | g \rangle = \langle g | f \rangle^* \]

(4) \[ \langle f | g_1 + c g_2 \rangle = \langle f | g_1 \rangle + c \langle f | g_2 \rangle \]

This shows that \( \langle f | g \rangle \) is an inner product on the space of complex valued continuous functions on \([a, b]\).
since these conditions imply the Cauchy-Schwarz inequality
\[ \|f - g\|_2^2 \leq \langle f, f \rangle \langle g, g \rangle \]
we have a norm
\[ \|f - g\|_2 = \int_a^b w(x) \left( f(x) - g(x) \right)^2 dx \]

One thing that is different than in the finite dimensional case is that Cauchy sequences do not always converge to vectors in the space.

\[ f(x) = \begin{cases} 0 & x < -\epsilon \\ \frac{1}{2\epsilon} & x \in (-\epsilon, \epsilon) \\ 1 & x > \epsilon \end{cases} \]

Consider
\[ \int_{-1}^1 \left| f(x) - f_{e^f}(x) \right|^2 dx < 2 (\text{larger of } \epsilon, \epsilon') \]

This clearly vanishes as the larger of \( \epsilon, \epsilon' \) goes to 0.
This is clearly a Cauchy sequence but it converges to the Heaviside function

\[ h(x) = \begin{cases} 
0 & x < 1 \\
1 & x > 1 
\end{cases} \]

which is a discontinuous function

Vector spaces where every Cauchy sequence converges to a vector in the space are called complete vector spaces.

This problem can be fixed - the first step is to generalize the notion of integral

1. A σ algebra is a collection of subsets \( S \) of \( X \) with the following properties:
   1. \( X \in S \)
   2. \( A \in S \rightarrow A^c \in S \)
   3. \( A \in S \rightarrow \bigcup_{n=1}^{\infty} A \in S \)

If \( S \) is a σ algebra in \( X \) then \( X \) is called a measurable space.
Consequences of the definition

\[(U A_i^c)^c\]
\[x \in (U A_i^c)^c \implies x \notin U A_i^c \implies x \in A_i \text{ for all } i\]
\[x \in \cap A_i\]

Countable unions and complements give countable intersection.

\[\text{Aim: if } S \text{ finite unions, finite intersection,}\]

\[A - B \quad x \in A \wedge x \notin B \quad x \in A \wedge x \in B^c \quad x \in A \cap B^c\]

(2) A function \(f\) (measure space) → vector space is measurable if the inverse image of any open set is measurable.

\[f^{-1}(0) = \{x \mid f(x) \in 0\}\]

(3) A function is continuous if the inverse image of any open set is open

\[f^{-1}(0,1) = 0\]

\[(x,y) < \delta \implies |f(x) - f(y)| < \epsilon\]

\[f^{-1}(0) = 0\]
A borel space is the smallest $\sigma$-algebra containing the open sets of $X$.

$O_{\text{open}} = O \in S$

A measure is a positive function on a $\sigma$-algebra that is countably additive.

$\mu(O) \geq 0$

$\emptyset \cap O_i = \emptyset$

$\sum \mu(O_i) = \mu(U O_i)$

Lebesgue measure

A measure on $\mathbb{R}^n$ that gives the volume of intervals and is translationally invariant.

$\mu_{\text{L}}([a,b]) = b-a$

$\mu_{\text{L}}((a,b]) = b-a$

$\mu_{\text{L}}((a,b)) = b-a$

We use this to define a new kind of integral.
\{ x \mid n \in \mathbb{N}, x \in [n,n+1) \} = \mathbb{N}

This is a measurable set if \( \delta(x) \) is continuous.

\[ \int f(x) \, dx = \lim_{n \to \infty} \sum_{n \in \mathbb{N}} \mu([x_{n}, x_{n+1}]) \]

where we define \( x_{0} = 0 \) when \( f \) is infinite on a set of measure \( c \).

When both integrals exist they give the area under the curve.

The Lebesgue integral is more general.

\[ f(x) = \begin{cases} 1 & \text{on rational } x \in [a,b] \\ 2 & \text{on irrational } x \in [a,b] \end{cases} \]

We need to find the volume of the rational number between \( a, b \).
These numbers have the sum
\[
\sum_{n=1}^{m} \frac{m}{n^3} = \left( \frac{m}{n} - \frac{1}{2n^3} \right) \frac{m}{n^3} + \frac{1}{2n^3}
\]

Consider
\[
\mathcal{A}_m = \left[ \frac{m}{n} - \frac{1}{2n^3} \right] \frac{m}{n^3} + \frac{1}{2n^3}
\]

\[
\mathcal{M} (A_m) = \frac{E}{n^3}
\]

We sum this over all rationals between \(a\) and \(b\) in each \(n\) under \( (b-a)n \) values of \(m\) so the total volume of the rationals is less than
\[
\frac{1}{n} \sum_{n=1}^{a} (b-a) n \frac{E}{n^3} = \frac{E (b-a)}{n^3} \sum_{n=1}^{a} \frac{1}{n^3}
\]

This can be made as small as desired. The volume of the complement of \(b-a\)
\[
\int_{a}^{b} f(x) dx = 2 \times (b-a)
\]

with the Lebesgue integral
\[
\int_{a}^{b} \omega(x) |f(x) - g(y)|^2 dx = 0
\]

only means that
\[
|f(x) - g(y)|
\]
differs from 0 on a set of Lebesgue measure 0.
The Riesz Fischer theorem asserts that if the inner product is defined using the Lebesgue measure that Cauchy sequences converge to a function in the space. In this case, the space of functions are Lebesgue measurable functions which contain continuous functions, but is a larger space.

Vectors in this case are equivalence classes of functions that differ on sets of measure 0.

Remark - the proof of the theorem uses the Cauchy sequence to construct a convergent sum, which is then shown to be measurable.