Phys 5742
Homework 1 - Due Friday 1/26

Problem 1: Show that if $U(\lambda)$ satisfies $U(\lambda_1 + \lambda_2) = U(\lambda_1)U(\lambda_2)$ then $U(\lambda)$ cannot be antiunitary.

Problem 2: Assume a vector $V$ is rotated about the $z$ axis (active rotation). To show that the components of the rotated $V$ are related to the components of the original $V$ by

$$
\begin{pmatrix}
V'_x \\
V'_y \\
V'_z
\end{pmatrix} =
\begin{pmatrix}
\cos(\theta) & -\sin(\theta) & 0 \\
\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
V_x \\
V_y \\
V_z
\end{pmatrix}.
$$

(1)

Problem 3: Differentiate

$$
e^{-iG\theta} V e^{iG\theta} = R^\dagger V =
\begin{pmatrix}
\cos(\theta) & \sin(\theta) & 0 \\
-\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
V_x \\
V_y \\
V_z
\end{pmatrix}
$$

(2)

with respect to $\theta$ and then set $\theta$ to zero to show:

$$- [iG, V] =
\begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
V_x \\
V_y \\
V_z
\end{pmatrix}.
$$

(3)

Component by component this gives

$$

(4)

where $G$ is the infinitesimal generator of rotations about the $z$ axis.

Problem 4: Show that a general $SU(2)$ matrix can be expressed in the form

$$W = e^{-i\frac{\theta}{2} \sigma} = \cos(\frac{\theta}{2})I - i\frac{\theta}{2} \sin(\frac{\theta}{2}).
$$

(5)

Problem 5: Use (5) to show that for

$$W = e^{-i\frac{\theta}{2} \sigma_x}
$$

(6)

that

$$R_{ij} = \frac{1}{2} \text{Tr}(\sigma_i W \sigma_j W^\dagger)
$$

(7)
gives $R$ corresponding to a rotation about the $\hat{z}$ axis:

$$
\begin{pmatrix}
\cos(\theta) & -\sin(\theta) & 0 \\
\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

**Problem 6:** Show that if a Hamiltonian is rotationally invariant, $U(\theta) H U^\dagger(\theta) = H$ then $[G, H] = 0$ and $G$ is a conserved quantity.