Problem 1: Use the relation

\[ Y_{lm}^{m}(\hat{n}) = \sqrt{\frac{1l + 1}{4\pi}} D_{m0}^{*l}(R) \]

where \( \hat{n} = R\hat{z} \) to integrate a product of three spherical harmonics

\[ \int \sin^7(\theta) d\theta \int_0^{2\pi} d\phi Y_{lm}^{ma}(\theta, \phi) Y_{lm}^{mb}(\theta, \phi) Y_{lm}^{mc}(\theta, \phi) \]

Express your answer in terms of Clebsch Gordan coefficients.

Problem 2: Assume that a Hamiltonian is invariant with respect to rotations about the \( x \) and \( y \) axes. Show that it must also be invariant with respect to rotations about the \( z \) axis.

Problem 3: Express the spherical harmonics \( Y_{lm}^2(\theta, \phi) \) in term of the Cartesian coordinates, \( x/r, y/r, z/r \). Convince yourself that each \( Y_{lm}^2 \) is a homogeneous polynomial of degree 2 in these quantities.

Problem 4: The spherical harmonics \( Y_{lm}^1(\hat{r}) \) are simultaneous eigenstates of \( L^2 \) and \( L_z \) with the eigenvalues \( 1(1 + 1) = 2 \) and \( m \). Use properties of rotations to find linear combinations of these states that are simultaneous eigenstates of \( L^2 \) and \( L_y \) with eigenvalues 2 and \( m \).

Problem 5: Assume that a spinless particle is bound to a rotationally invariant potential and assume that it is in an eigenstate of \( L^2 \) with eigenvalue \( 2(2+1) = 6 \). Show that this state must be degenerate. (this means that there is more than one eigenstate with the same energy eigenvalue).

Problem 6: Using the \( |n_+, n_-\rangle \) basis for the angular momentum states find operators (in terms of \( a_+ \) and \( a_\perp \) that raise and lower the eigenvalue \( j \) without changing \( m \)?)