1. Assume that an electron scatters off of a potential due to a spherically symmetric electric charge density, $-e_\rho(r)$. Find the scattering amplitude and differential cross section in the Born approximation. Show how these are related to the Fourier of this charge distribution.

2. The $S$ operator must be unitary. One way to ensure this is to write

$$S = \frac{I - iK}{1 + iK}$$

where $K = K^\dagger$. Find the relation between matrix elements of $K(E)$ and matrix elements of the transition operator $T(E + i\epsilon)$. Find an integral equation for $\langle k'|K(E)|k \rangle$ in terms of the potential. How is $K(E)$ related to the phase shift?

3. Calculate the differential cross section (in terms of the transition operator matrix elements) for two-body scattering in the laboratory frame where particle 1 of mass $m_1$ is initially at rest, particle 2 of mass $m_2$ is initially moving with momentum $\mathbf{p} = \mathbf{p}_2$ and the angular distribution of particle 1 is measured. (Hint - you need to find the initial relative velocity in this frame and integrate over all variables that are not measured.

4. Consider scattering of a neutron off of a proton (with radius of about $10^{-15}m$). For a 10 MeV beam of neutrons, how many partial waves would you need to accurately describe this scattering reaction?

5. Find the $l = 0$ partial wave scattering wave function $\langle r, 0, 0|k^-, 0, 0 \rangle$ for the delta shell potential, $V(r) = -V_0\delta(r - R)$.

6. Find the $l = 0$ scattering amplitude for the potential of problem 5.
\[
\frac{\partial g}{\partial \theta} = 1 F 1^2
\]

\[
F = -(2\pi)^2 \mu \hbar \langle \tilde{r}' \mid V \mid \tilde{r} \rangle \\
= -(2\pi)^2 \mu \hbar \left\{ -i \frac{\tilde{r}' \cdot \tilde{r}}{2\pi \hbar^3} \right\} \int \frac{e^{-i \tilde{r}' \cdot \tilde{r}}}{|\tilde{r}' - \tilde{r}|} d^3 r' d^3 r
\]

Let \( \tilde{r}'' = \tilde{r} - \tilde{r}' \) \quad \text{and} \quad d^3 r'' = d^3 r

\[
= \frac{(2\pi)^2 e^{i \mu \hbar}}{(2\pi \hbar)^3} \int e^{-i (\tilde{r}'' - \tilde{r}) \cdot (\tilde{r}'' + \tilde{r}')/\hbar} \frac{\rho(\tilde{r}')}{|\tilde{r}' - \tilde{r}|} d^3 r'' d^3 r'
\]

\[
= \frac{e^{i \mu \hbar}}{(2\pi \hbar)^3} \int \rho(\tilde{r}) e^{-i (\tilde{r} - \tilde{r}'') \cdot \tilde{r}''/\hbar} \int \frac{e^{-i (\tilde{r} - \tilde{r}'') \cdot \tilde{r}''}}{(2\pi \hbar)^3} d^3 r''
\]

If we define \( \tilde{Q} = \tilde{r} - \tilde{r}' \) the first integral is the Fourier transform of the potential \( \tilde{\rho}(\tilde{Q}) \).

The second term is

\[
(2\pi \hbar)^{3/2} \times \left[ -(2\pi)^2 \mu \hbar \right] \langle \tilde{r}' \mid \left( -\frac{e}{\tilde{r}} \right) \mid \tilde{r} \rangle
\]

\text{Coulomb scattering amplitude in the Born Approximation}

\[
F = (2\pi \hbar)^{3/2} \tilde{\rho}(\tilde{Q}) F_{\text{coul}}(\tilde{Q})
\]

point charge
\[ S = \frac{1 - iK}{1 + iK} \quad S = e^{2i\delta} \]

First note
\( (1 + iK)S = 1 - iK \)
\( S - 1 = -iK(1 + S) \)
\( K = i \frac{S - 1}{S + 1} = i \frac{e^{2i\delta} - 1}{e^{2i\delta} + 1} - i \frac{e^{i\delta}}{e^{i\delta}} \left( \frac{e^{i\delta} - e^{-i\delta}}{e^{i\delta} + e^{-i\delta}} \right)^{1/2} \)
\[ = -\frac{\sin \delta}{\cos \delta} \]

as an operator
\[ K = -\tan \delta \]

Relation with \( T \) - In the energy basis:
\[ |\bar{E}\rangle = |R \theta \phi \rangle \quad |E \bar{K}\rangle = |E \theta \phi \rangle \]
\[ |R \theta \phi \rangle \sqrt{\frac{2}{\kappa^2 r_0^4}} = |R \theta \phi \rangle k u = |E \theta \phi \rangle \]

In the \( E \) basis:
\[ S - 1 = -iK(1 + S) \]
\[ \gamma = 2\pi iT \]
\[ -2\pi iT = -iK(1 + 2\pi iT) \]
\[ = -2iK - 2\pi KT \]
\[ T = \frac{1}{\hbar} K \quad \frac{1}{\hbar} KT = \frac{1}{\hbar} K + iKT \]
\[ T = \frac{1}{\hbar} K + iKT \]
as an equation in the energy basis
this has the form
\[ \langle \vec{k}' | \Sigma | \vec{k} \rangle = \frac{1}{\pi} \langle \vec{k}' | \Sigma | \vec{k} \rangle \\
+ i \int \langle \vec{k}' | \Sigma | \vec{k} \rangle \, d\Omega(k) \langle \vec{k}' | \Sigma | \vec{k} \rangle \]
changing to the \( k' \) basis gives
\[ \langle \vec{k}' | \Sigma | \vec{k} \rangle = \frac{1}{\pi} \langle \vec{k}' | \Sigma | \vec{k} \rangle \\
+ i \int \langle \vec{k}' | \Sigma | \vec{k} \rangle \, d\Omega(k) \langle \vec{k}' | \Sigma | \vec{k} \rangle \]
\[ \langle \vec{k}' | \Sigma | \vec{k} \rangle = \frac{1}{\pi} \langle \vec{k}' | \Sigma | \vec{k} \rangle + i \int \langle \vec{k}' | \Sigma | \vec{k} \rangle \, d\Omega(k) \langle \vec{k}' | \Sigma | \vec{k} \rangle \]

Note with partial waves this becomes an algebraic equation.

To relate \( k' \) to \( \Sigma \) it is useful to consider the constraint equations

\[ (1) \quad T = \frac{1}{\hbar} K + i K \otimes T \quad S = S(E-H_0) \]
\[ (2) \quad T = V + V \otimes T \quad \bar{R} = (E-H_0)^T \]

We want to eliminate \( T \). It is useful to express the first equation as

\[ (1') = T = \frac{1}{\hbar} K + \frac{1}{\hbar} K (i \otimes S) T \]
and the second equation is

\[ T = V + V (R - i n s + i n s) T \]

\[ (1 - V (R - i n s)) T = V + V i n s T \]

\[ T = (1 - V (R - i n s))^T V + (1 - V (R - i n s))^T V i n s T \]

Comparing these equations:

\[ \frac{1}{n} K = (1 - V (R - i n s))^T V \alpha \]

\[ \frac{1}{n} K - \frac{1}{n} V (R - i n s) K = V \]

\[ \alpha \]

\[ K = \pi V + V (R - i n s) K \]

Remark: \( \frac{1}{E - H + i \epsilon} \) is called the principal value of \( \frac{1}{E - H + i \epsilon} \)
\[ d\sigma = \frac{(2\pi)^4 \hbar^2}{(P_2/m)_1} K P_1 P'_1 \| T(E+i\epsilon) \| K P_2 > \| P_2' > \| dP_1 dP'_1 \times \delta(E-E') \delta(P-P') \]

In the lab frame \( \bar{p}_1 = 0 \) \( \bar{p}_2 = m_2 \nu \)

\[ E'_1 - E = \frac{P_1^{12}}{2m_1} + \frac{P_2^{12}}{2m_2} - \frac{P_2^2}{2m_2} \quad \bar{p}_1 + \bar{p}_1' - \bar{p}_2 = 0 \]

If we want to measure \( \bar{p}_2' \) integrally over \( \bar{p}_1' \) this eliminates the

\[ \delta(\bar{p}_1' + \bar{p}_2' - \bar{p}_1) \]

\[ E'_1 - E = \frac{P_2^{12}}{2m_2} + \frac{P_2^{12}}{2m_1} + \frac{(P_2 - P_2')^2}{2m_1} \]

\[ = \frac{P_2^{12}}{2m_2} + \frac{P_2^{12}}{2m_1} - \frac{P_2}{2m_2} + \frac{P_2}{2m_1} - \frac{P_2' P_2}{m_1} = 0 \]

\[ \frac{dE'_1}{dP_2'} = \left( \frac{P_2}{m_2} + \frac{P_2}{m_1} - \frac{P_2 \cos \theta}{m_1} \right) = \left( \frac{P_2}{m_2} - \frac{P_2 \cos \theta}{m_1} \right) \]

\[ \frac{dP_2'}{dE'_1} = m_2 \frac{m_1}{P_2' - P_1' \cos \theta} = \frac{m_2}{m_1 + m_2} \frac{m_1 + m_2}{P_1' - m_2 P_2 \cos \theta} \]

\[ = \frac{m_2 m_1}{(m_1 + m_2) P_1' - m_2 P_2 \cos \theta} \]

\[ \frac{d\sigma}{d\Omega(P'_1)} = \frac{(2\pi)^4 \hbar^2}{(P_2/m)_1} K P_1 P'_1 \| T(E+i\epsilon) \| 0, P_2 > \| dP_1 \times \delta(E-E') \delta(P-P') \]

\[ \frac{d\sigma}{d\Omega(P'_1)} = \frac{(2\pi)^4 \hbar^2 m_2^2 m_1}{(m_1 + m_2) P_2 P'_1 - P_2' \cos \theta} \]
Classically the maximum orbital momentum is about
\[ R = 10^{-15} \text{ m} = 1 \text{ fm} \]

\[ \beta = \frac{10 \text{ MeV/c}}{197.1 \text{ MeV/c}} = 0.05 \]

This suggests \( l = 0 \) or \( l = 1 \) should be sufficient to treat scattering at this momentum.

\[ \langle R_0 k \rangle = \frac{4\pi}{(2\pi)^{3/2}} J_0\left(\frac{kr}{\hbar}\right) - \frac{4\pi^3}{(2\pi)^3} \pi \hbar \int J_0\left(\frac{kr}{\hbar}\right) h_0\left(\frac{kr}{\hbar}\right) \]

\[ \times \left(-V_0 R\right) \delta\left(r-R\right) \left(\frac{k}{\hbar}\right)^2 \langle R_0 k \rangle \]

\[ = \frac{4\pi}{(2\pi)^{3/2}} J_0\left(\frac{kr}{\hbar}\right) + \frac{4\pi^3}{(2\pi)^3} \pi \hbar \int V_0 R^3 \left(\frac{kr}{\hbar}\right)^2 h_0\left(\frac{kr}{\hbar}\right) \langle R_0 k \rangle \]

Set \( r = R \) to find \( \langle R_0 k \rangle \)

\[ \langle R_0 k \rangle = \frac{4\pi}{(2\pi)^{3/2}} J_0\left(\frac{kr}{\hbar}\right) + \frac{4\pi^3}{(2\pi)^3} \pi \hbar \int V_0 R^3 \left(\frac{kr}{\hbar}\right)^2 h_0\left(\frac{kr}{\hbar}\right) \]

\[ \langle R_0 k \rangle = \frac{4\pi}{(2\pi)^{3/2}} J_0\left(\frac{kr}{\hbar}\right) \]

\[ 1 - \frac{4\pi^2 \hbar^2 V_0 R^3}{(2\pi)^3} J_0\left(\frac{kr}{\hbar}\right) h_0\left(\frac{kr}{\hbar}\right) \]
Note that \( J_0(x) = \frac{\sin x}{x} \) and \( h^n_0(x) = \frac{e^{ix}}{x} \).

Using these, note

\[
\langle R_{10} R_0 \rangle = \frac{4\pi}{(2\pi\hbar)^3} \left( \frac{\sin \left( \frac{KR}{\hbar} \right)}{KR} \right) \frac{k^2 R^2}{(2\pi)^3} \sin \left( \frac{RR'}{\hbar} \right) e^{i\frac{RR'}{\hbar}}
\]

\[
= \frac{4\pi}{(2\pi\hbar)^3} \frac{k}{KR} \sin \left( \frac{KR}{\hbar} \right) \frac{1}{1 - \frac{2\hbar V_0 R}{\pi R} \sin \left( \frac{RR'}{\hbar} \right) e^{i\frac{RR'}{\hbar}}}
\]

Using this in the solved form

\[
\langle R_{10} R_0 \rangle = \frac{4\pi}{(2\pi\hbar)^3} \left[ \frac{k}{KR} \sin \left( \frac{KR}{\hbar} \right) \frac{1}{1 - \frac{2\hbar V_0 R}{\pi R} \sin \left( \frac{RR'}{\hbar} \right) e^{i\frac{RR'}{\hbar}}} \right]
\]

To find the \( R=0 \) scattering amplitude

\[
F = -\left(2\pi\right)^2 \mu k^2 \langle k | t_0 | k' \rangle
\]

\[
= -\left(2\pi\right)^2 \mu k^2 \langle k' | V | k \rangle
\]

\[
= -\left(2\pi\right)^2 \mu k^2 \int \langle R_{10} R_0 \rangle V(r) r^2 dr \langle R_{01} R_0 \rangle
\]

\[
= -\left(2\pi\right)^2 \mu k^2 \left( -V_0 R^3 \right) \frac{4\pi i^6}{(2\pi\hbar)^3} J_0(kR\hbar) \langle R_{01} R_0 \rangle
\]
From problem 5

\[ I = \frac{(2\pi)^2 4\pi V_0 R^3 u R}{(2\pi \eta)^3 \lambda} \frac{4\pi}{(2\pi \eta)^3 \lambda} \frac{1}{2} \left( \frac{RR}{\eta} \right) \delta_{-1} \left( \frac{RR}{\eta} \right) \frac{\eta}{R^n} \]

\[ I = \frac{8\pi V_0 R u}{R^2} \frac{\sin^2 \left( \frac{RR}{\eta} \right)}{1 - \frac{2uvR}{\pi R} \sin \left( \frac{RR}{\eta} \right) e^{iRR/\eta}} \]