Lecture 1

See notes on rotations in syllabus outline

* Unitary I parameter groups
* Rotations about a fixed axis
* SU(2)
* Angular momentum
* Representations of SU(2)

**Definition:** A collection of unitary operators $U(\lambda)$ is called a unitary parameter group if $U(\lambda)$ satisfies the following conditions:

1. $U(0) = I$
2. $U(\lambda_2)U(\lambda_1) = U(\lambda_2 + \lambda_1)$
3. $U(\lambda)^+ = U(\lambda)^\dagger = U(-\lambda)$

**Examples:**

1. Rotations about a fixed axis
2. Time evolution
3. Translations in a given direction
Theorem: If $U(x)$ is a unitary
1 parameter group then

$$U(1) = e^{-i\lambda}$$

where $G = G^*$; $G$ is independent
of $x$ and

$$e^{i\lambda} = I + \sum_{n=1}^{\infty} \frac{(i\lambda)^n}{n!}$$

The operator $G$ is called the
infinitesimal generator of $U(x)$

Proof:

$$G = \frac{d}{d\lambda} I = \frac{d}{d\lambda} (U(x)U^+(x)) = \frac{dU}{d\lambda} (x)U^+(x) + U(x) \frac{dU^+}{d\lambda} (x)$$

This gives

$$\frac{dU}{d\lambda} (x)U^+(x) = -U(x) \frac{dU^+}{d\lambda} (x) = -(\frac{dU}{d\lambda} (x)U^+(x))^+$$

define

$$G(x) = i \frac{dU}{d\lambda} (x)U^+(x)$$

Then the above equation gives

$$iG(x) = - (iG(x))^+ = iG^+(x)$$
or \( G(\lambda) = G(\lambda)^+ \)

To show \( G(\lambda) \) is independent of \( \lambda \) let \( \lambda' = \lambda + c \) where \( c \) is a constant,

\[
\frac{d}{d\lambda'} = \frac{d}{d\lambda} \frac{d}{d\lambda'} = \frac{d}{d\lambda'} (\lambda' \cdot c) \frac{d}{d\lambda} = \frac{d}{d\lambda}
\]

\[
\frac{d}{d\lambda} \ U(\lambda) \cdot U^+(\lambda) = \frac{d}{d\lambda} \ U(\lambda' \cdot c) \cdot U^+(\lambda' \cdot c) =
\]

\[
\frac{d}{d\lambda'} \ U(\lambda') \cdot U^+(\lambda') \cdot U(\lambda') \cdot U^+(\lambda') \cdot U(\lambda') \cdot U^+(\lambda') \cdot U(\lambda') \cdot U^+(\lambda') = \frac{d}{d\lambda}
\]

This implies \( G(\lambda) = G(\lambda') \)

Using the definition

\[
G = i \frac{dU}{d\lambda} \cdot U^+(\lambda) = i
\]

\[
\frac{dU}{d\lambda} = -i \sigma U(\lambda)
\]

Differentiating \( n \) times

\[
\frac{d^n U}{d\lambda^n} = (-i)^n \sigma^n U(\lambda)
\]
we can use these to construct the Taylor series
\[ e^{-i\sigma} = 1 + \sum_{n=1}^{\infty} \frac{(-i\sigma)^n}{n!} n^n \]

Note
\[ \frac{d}{d\lambda} (e^{-i\sigma}) = \sum_{n=1}^{\infty} \frac{n x^{n-1} (-i\sigma)^n}{n!} = -i \sigma \sum_{n=1}^{\infty} \frac{x^{n-1} (-i\sigma)^n}{(n-1)!} \]
let \( m = n-1 \)
\[ = -i \sigma \sum_{m=0}^{\infty} \frac{(-i\sigma)^m}{m!} 6^m = -i \sigma e^{-i\sigma} \]

The series satisfies the differential equation and initial condition

(2) converges when applied to superposition of eigenstates of 6 with maximum eigenvalue finite (this is a dense set of vectors)

Rotations about a fixed axis

In a quantum theory if we do an experiment in a rotated laboratory (isolated system) we expect to get identical quantum probabilities
\[ |\psi\rangle \rightarrow |\psi'\rangle \]
\[ |\phi\rangle \rightarrow |\phi'\rangle \]
\[ P = \langle \psi | \phi |^2 = \langle \psi' | \phi' |^2 \]

\[ L \]
\[ 4 \]
If this is true for every \( |\psi\rangle \) Wigner's theorem (see notes) implies either

\[
|\psi'\rangle = |\psi\rangle \quad \text{(unitary)} \\
|\psi'\rangle = i|\psi\rangle \quad \text{(antiunitary)}
\]

The nontrivial part is to show that these are the only possibilities.

For rotations about a fixed axis:

\[
|\psi''\rangle = T(0,1)T(0,2)T(0,1)|\psi\rangle
\]

which means

\[
T(0,2)T(0,1) = T(0,1+0,2)
\]

\( i |\psi(0,0)\rangle \)

We show that \( T(0) \) with this property cannot be antiunitary.

In most cases the possible phase factor can be eliminated by redefining \( T(0) \) — the nontrivial exception is transforming that shift constant velocity:

\[
\vec{x} \to \vec{x}' = \vec{x} - \vec{v}t
\]
It then follows that for rotations about a given axis

\[ |\psi\rangle \rightarrow |\psi'\rangle = U(\theta) |\psi\rangle \]

where

\[ U(\theta) \text{ is unitary} \]

\[ U(\theta_2)U(\theta_1) = U(\theta_1 + \theta_2) \]

\[ U(0) = I \]

This means for a rotation about a fixed axis \( U(\theta) \) is a 1 parameter unitary group. This means

\[ U(0) = e^{-i0\theta} \]

\( e \) is called the infinitesimal generator of rotations about the given axis.

We can treat rotations as active or passive:

- **Passive**: vector is unchanged, basis is transformed
- **Active**: basis is fixed, vector changes
Following the text I will consider active transformations.

Let $\vec{\nabla}$ be a vector operator, this means $\vec{\nabla} = \vec{V}_x \hat{x} + \vec{V}_y \hat{y} + \vec{V}_z \hat{z}$ when each component of $\vec{\nabla}$ is an operator ($\vec{\nabla}_x$, $\vec{\nabla}_z = -i \vec{\nabla}_x$, etc.)

and the basis is fixed

\[ \begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \]

\[ = \sum_{i<j} R_{ij} V_j \]

If $\vec{\nabla}$ is an operator

$U(R) \vec{\nabla} U^\dagger(R) = R^\dagger \vec{\nabla}$

why $R^\dagger$

$U(R_2) U(R_1) \vec{\nabla} U^\dagger(R_1) U^\dagger(R_2) = \ldots$
\[ U(R_z) R_1^T \tilde{V} U(R_2^T) = \mathcal{S} R_{1ij}^T U(R_z) \tilde{V}_j U(R_2^T) \]
\[ = \mathcal{S} R_{1ij} R_{2jk} V_k \mathcal{S} (R_2 R_1)^T i_j V_j \]

(without the transpose the rotations come in the wrong order)

\[ e^{-i \theta} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_y \\ V_z \\ V_x \end{pmatrix} \]

homework: \[ \frac{d}{d\omega} | \omega = -1 \]

\[ -i \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_y \\ -V_x \\ 0 \end{pmatrix} \]

which gives:

\[ [G, V_x] = i V_y \]
\[ [G, V_y] = -i V_x \]
\[ [G, V_z] = 0 \]

(i.e., for a positive rotation \((ccw)\) about the \(z\) axis)
This exercise can be repeated for rotations about the $x$ any $y$ axes:

\[
\begin{align*}
[ G_x V_x ] &= 0 \\
[ G_y V_y ] &= i V_2 \\
[ G_x V_2 ] &= -i V_y \\
[ G_y V_3 ] &= -i V_2 \\
[ G_y V_4 ] &= 0 \\
[ G_y V_5 ] &= i V_x \\
\end{align*}
\]

These can be summarized if we let $V_1 = V_x$, $V_2 = V_y$, $V_3 = V_z$, and

$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = -\epsilon_{321} = -\epsilon_{132} = 1$

0 otherwise.

\[
[ G_i V_j ] = i \sum_{k=1}^{3} \epsilon_{ijk} V_k
\]

These relations are true for any operator that transforms like a vector under rotations.
If we have a Hamiltonian that is invariant with respect to rotations

\[ U(0) H U(0)^\dagger = H \]

Exercise: show

\[ [\theta, H] = 0 \]

which means that \( \theta \) is a conserved quantity. The conserved quantity for rotationally invariant systems is the angular momentum

\[ \mathbf{\hat{\theta}} = i \hbar \frac{\partial}{\partial \theta} \]

since \( U(0) = e^{-i \theta \hat{\theta}} \). \( \theta \) must be dimensionless for this to make sense, since \( \theta \) is dimensionless.

\[ \theta = \frac{J_z}{\hbar} \]

where \( \hbar \) has units of angular momentum.

In what follows, I use units where \( \hbar = c = 1 \).

\[ G_i = J_i / \hbar \]
SU(2)

It is sometimes useful to represent vectors by $2 \times 2$ traceless Hermitian matrices rather than by their components:

$$X = \bar{x} \cdot \mathbf{g} = \begin{pmatrix} x^3 & x' + i x_2 \\ x' + i x_2 & -x^3 \end{pmatrix}$$

$$\bar{x} = \frac{1}{2} \text{Tr} (\mathbf{g} X)$$

Properties of $X$

$$X = X^\dagger \quad \text{(this is equivalent to saying } \bar{x} = x^*)$$

$$\text{Tr} (X) = 0$$

$$\det (X) = -\bar{x} \cdot x = -\|\bar{x}\|^2$$

These properties are preserved under transformations of the form

$$X \rightarrow X' = W X W^\dagger = W X W^\dagger$$

where $WW^\dagger = I$ and $\det W = 1$

$$\det X' = \det W \det X \det W^\dagger$$
\[ \det W = \exp(i\theta), \quad \det(W^t) = \exp(-i\theta) \text{ and the phases cancel.} \]

Also note \( X \rightarrow -X \) corresponds to a space reflection, which cannot be represented by \( U X U^t \).

Finally,
\[ X' = W X W^t = (-W) X (-W)^t \]

where both
\[ \det(W) = \det(-W) = 1 \]

From this we conclude
\[ X' = W X W^t \]
corresponds to a rotation, and there are at least 2 \( U \)'s for a given 3x3 rotation.

\[ X' = \frac{1}{2} \text{Tr}(\bar{\sigma} X) = \]
\[ = \frac{1}{2} \text{Tr}(\bar{\sigma} W \bar{\sigma} \bar{X} W^t) \]
\[ = \frac{1}{2} \text{Tr}(\bar{\sigma} W \bar{X} W^t) \bar{x} i \]
\[ = RX \]
Homework:
show that a general SU(2) matrix can be expressed as
\[ W = e^{-\frac{i}{2} \theta \hat{z} \cdot \vec{c}} = \cos \left( \frac{\theta}{2} \right) I - i \hat{\vec{c}} \cdot \sin \left( \frac{\theta}{2} \right) \]

Homework:
show that a general SU(2) matrix can be expressed as
\[ W = e^{i \theta \vec{r} \cdot \vec{e}} \]
where \( \vec{e} \cdot \vec{e} = 1 \)

Homework:
show that \( W = e^{-\frac{1}{2} \theta \hat{z} \cdot \vec{c}} \)
\[ R_{ij} = \frac{1}{2} \text{Tr} (\sigma_i W \sigma_j W^\dagger) \]
corresponds to a rotation about the z axis through an angle \( \theta \).

Angular Momentum

A general angular momentum operator satisfies
\[ \{ \hat{J}_i, \hat{J}_j \} = i \sum_k \epsilon_{ijk} \hat{J}_k \]
Homework: show that the commutation relations imply

\[ [J_i, \tilde{J}^2] = 0 \]

where \( \tilde{J}^2 = J_1^2 + J_2^2 + J_3^2 \).

This means that it is possible to find simultaneous eigenvectors of \( \tilde{J}^2 \) and \( J_2 \) i.e., \( |m\mu\rangle \)

\[ \tilde{J}^2 |m\mu\rangle = n |m\mu\rangle \]
\[ J_2 |m\mu\rangle = \mu |m\mu\rangle \]

(usually the RHS would be multiplied by \( \hbar^2 \) or \( \hbar \))

Define \( \tilde{J}_\pm = J_x \pm i J_y \)

Homework: show that the angular momentum commutation relation imply

\[ [J_2, \tilde{J}_\pm] = \pm \tilde{J}_\pm \]

This means

\[ J_2 \tilde{J}_\pm |m\mu\rangle = (\tilde{J}_\pm J_2 \mp J_\pm) |m\mu\rangle = \]
\[ \langle \mu \pm 1 \rangle J_\pm |\mu \rangle \]

This means that \( J_\pm |\mu \rangle \) is an eigenstate of \( J^2 \) and \( J_z \) with eigenvalue \( \mu \pm 1 \).

**Exercise:** Show

\[ J_+ J_- = J^2 - J_z^2 \pm J_z \]

given this we can compute the norm of \( J_\pm |\mu \rangle \)

\[ ||J_\pm |\mu \rangle||^2 = \langle \mu \mu | J_+ J_- |\mu \rangle = \langle \mu \mu | J^2 - J_z^2 \pm J_z |\mu \rangle = (\mu - \mu(\mu \pm 1)) = \mu - \mu(\mu \pm 1) \]

Note that the norm of a vector must be positive.

For fixed \( \mu \) if we keep increasing or decreasing \( \mu \) the expression above eventually becomes negative.

This means that there must be a maximum and minimum value of \( \mu \) satisfying...
\[ \nabla^+ \left( \text{Im} \ U_{\text{max}} \right) = \nabla^+ \left( \text{Im} \ U_{\text{min}} \right) = 0 \]

This requires

\[ M - U_{\text{max}}(U_{\text{max}} + 1) = \]
\[ \eta - U_{\text{min}}(U_{\text{min}} - 1) = 0 \]

These are quadratic equations for \( U_{\text{max}} \) in terms of \( U_{\text{min}} \)

\[ U_{\text{min}}^2 - U_{\text{min}} - U_{\text{max}}(U_{\text{max}} + 1) = 0 \]

\[ U_{\text{min}} = \frac{1}{2} \left( 1 \pm \sqrt{1 + 4U_{\text{max}}^2 + 4U_{\text{max}}} \right) \]

\[ = \frac{1}{2} \left( 1 \pm (1 + 2U_{\text{max}}) \right) \]

\[ = \begin{cases} 
\frac{1}{2} (2) (1 + U_{\text{max}}) = U_{\text{max}} + 1 \\
\frac{1}{2} (2)(-U_{\text{min}}) = -U_{\text{max}} 
\end{cases} \]

For the first root \( U_{\text{min}} > U_{\text{max}} \) so it is not the correct solution. Thus given

\[ U_{\text{min}} = -U_{\text{max}} \]

\[ \eta = U_{\text{max}}(U_{\text{max}} + 1) \]
\[ = U_{\text{min}}(U_{\text{min}} - 1) \]
It is customary to call
\[ \Delta = \Delta \max \]
\[ n = \frac{\Delta}{\Delta + 1} \]
we relabel the states by
\[ | \mu \rangle \rightarrow \equiv | \mu \rangle \]
\[ J_\pm | \mu \rangle = \mu | \mu \rangle \]
\[ J_\pm | \mu \rangle = \frac{\Delta (\Delta + 1)}{\Delta (\Delta + 1) + 1} | \mu \rangle \]
recall
\[ \langle \eta \mu | J_\mp J_\pm | \eta \mu \rangle = \]
\[ \eta - \mu + 1 = \]
\[ \frac{\Delta (\Delta + 1)}{\Delta (\Delta + 1) - \mu (\mu + 1)} = \]
\[ (\Delta + 1)(\Delta + 1) \]
this means
\[ J_\pm | \mu \rangle = \sqrt{\Delta (\Delta + 1)} | \mu \pm 1 \rangle \]
\[ = \sqrt{\Delta (\Delta + 1) - \mu (\mu + 1)} | \mu \pm 1 \rangle \]