Lecture 2

Density matrices and quantum ensembles

\[ \rho = \bigoplus \langle a_n | a_n \rangle P_n \]

\[ \sum P_n = 1 \quad \text{(probability of finding } | a_n \rangle \text{ in beam)} \]

\[ \langle a_n | a_n \rangle = 1 \quad \text{but } | a_n \rangle \text{ do not have to be orthonormal} \]

Properties

1. Traces: \{ | b_n \rangle \} orthonormal basis

\[ \text{Tr } \mathbf{0} = \sum_{n=0}^{\infty} \langle b_n | \mathbf{0} | b_n \rangle \]

Last time we showed that the trace of an operator is independent of basis.

2. \[ \rho^* = \rho \quad \rho^* = \sum_{n} (\langle a_n | a_n \rangle^* P_n^* \]

\[ = \sum_{n} | a_n \rangle \langle a_n | P_n \]

so \( \rho \) is Hermitian
\(\text{Tr } \rho = 1\)

\[
\text{Tr } \rho = \sum \langle b_m | a_n \rangle \langle a_{n^1} | P_n | b_m \rangle \\
= \sum \langle a_{n^1} b_m \rangle \langle P_m | a_n \rangle P_n \\
= \sum \langle a_n | b_n \rangle P_n \\
= \sum P_n = 1
\]

since \(P_n\) is a probability

\(\rho \succeq 0\)

\[
\langle v_1 | \rho | v_1 \rangle = \sum \langle v_1 | a_n \rangle P_n \langle a_{n^1} | v_1 \rangle \\
= \sum \langle v_1 | a_n \rangle | \langle a_n | v_1 \rangle |^2 P_n \\
\geq 0
\]

(this means that \(\rho\) can only have non-negative eigenvalues)

\(\text{Tr } \rho^2 \leq 1\)

since \(\rho = \rho^\dagger\), we can diagonalize \(\rho\)

\[
\rho = \sum \langle P_n | \rho | P_n \rangle P_n \rho P_n \rho
\]
In this case $1_{p_n}$ can be chosen as orthonormal since $\rho = \rho^+$

$p_n \geq 0$ since $\rho \geq 0$

$\sum p_n = 1$ since $\text{Tr} \rho = \sum \langle p_n \mid p_n \rangle p_n = \sum p_n$

$11$ basis independent

$\text{Tr} (\rho^2) = \text{Tr}\left(\sum (1_{p_n} \sum_{m<n} \langle p_n \mid p_m \rangle p_m \langle p_m \mid p_n \rangle)\right)$

$= \sum \langle b_n \mid p_n \rangle p_n^2 \langle p_n \mid b_n \rangle$

$= \sum_{n} \langle p_n \mid b_n \rangle \langle p_n \mid p_n \rangle p_n^2$

$= \sum \langle p_n \mid p_n \rangle p_n^2$

$= \sum p_n^2$

If $p_n > 0$ and $\sum p_n = 1$ then each $p_n \leq 1$ $p_n^2 \leq p_n$ $\sum p_n^2 \leq \sum p_n$ $1$

If $\text{Tr} (\rho^2) = 1$ then this means for one value of $n$ $p_n = 1$ for all others $p_n = 0$
If $\text{Tr} \rho^2 = 1$, $\rho$ is called a pure state. It density matrix has the form

$$\rho = 1 |a\rangle \langle a|$$

where $\langle a|a\rangle = 1$

If $\text{Tr} \rho^2 < 1$, $\rho$ is called a mixed state.

Q: Why is $\rho$ called a state?

Consider a probability distribution of expectation values of an operator $\Omega$

$$\sum_n p_n \langle a_n | \Omega | a_n \rangle$$

This is called an ensemble average. If we define

$$\rho = 1 |a_n\rangle \langle a_n| p_n$$

Then

$$\text{Tr} (\rho \Omega) = \sum_n <a_n | \Omega | a_n > p_n <a_n | a_n | b_m >$$

$$= \sum_n <a_n | a_n | b_m > <b_m | a_n> p_n$$

$$= \sum_n <a_n | a_n | a_n > p_n$$
This means that the $\text{Tr}(\rho \Theta)$ is what you get by measuring $\Theta$ many times in an ensemble of states.

Note - The probability is classical - if $\Theta$ can be expanded in terms of eigenstates

$$\Theta = \sum \ket{\Theta_n} \bra{\Theta_n}$$

Then

$$\text{Tr}(\Theta \rho) =$$

$$\sum \bra{b_n} \Theta_n \ket{\Theta_n} \bra{\Theta_n} \rho \ket{\Theta_n}$$

$$\sum \bra{\Theta_n} \rho \ket{\Theta_n} \bra{\Theta_n} \Theta_n$$

$$\sum \left| \bra{\Theta_n} \rho \ket{\Theta_n} \right|^2 \rho \Theta_n$$

quantum probability → classical probability of measuring state $\Theta_n$ being in state $\Theta_n$ beam
Polarization vectors

Homework: show the pauli matrices and the identity \( \sigma_0 \)

\[
\sigma_0^+ = \sigma_0, \quad \sigma_i^+ = \sigma_i;
\]

\( \text{Tr} (\sigma_i) = 0 \quad \text{Tr} (\sigma_0 \sigma_b) = 2 \delta_{ab} \)

Since a 2x2 density operator is hermitian it can in general be expressed as

\[ \rho = \sum_i p_i \sigma_i \]

\[ \text{Tr} \rho = \sum_i p_i \text{Tr} (\sigma_i) = p_0 \cdot 2 \]

Since \( \text{Tr} \rho = 1 \), we must have \( p_0 = \frac{1}{2} \)

\[ \rho^2 = \sum_i p_i p_k \sigma_i \sigma_k \]

\[ \text{Tr} \rho^2 = \sum_i p_i p_k \text{Tr} (\sigma_i \sigma_k) = 2 \sum p_i^2 \]

\[ = 2 \cdot \left( \frac{1}{2} \right)^2 + 2 \bar{p}^2 \leq 1 \]

\[ 2 \bar{p}^2 \leq 1 - \frac{1}{2} = \frac{1}{2} \quad \bar{p}^2 \leq \frac{1}{4} \quad 1 \bar{p}_i \leq \frac{1}{2} \]

\[
\rho = \frac{1}{2} (I + \bar{p} \cdot \sigma)
\]

\( \bar{p} \) is called the polarization vector.
\[ \bar{\rho} = \hat{\rho} \]

\[
\rho = \frac{1}{2} \begin{pmatrix} \sigma_0 + \sigma_3 \\ \sigma_0 + \sigma_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

If \( \bar{\rho} \) is a unit vector then \( \rho \) represents a pure state.

For a general \( \rho \)

\[
\text{Tr} (\rho \sigma) = \text{Tr} (\frac{1}{2} \sigma + \sigma \cdot \bar{\rho} \sigma)
\]

\[
= 0 + \frac{1}{2} \sum_{i} \rho_{i} \text{Tr} (\sigma_{i} \sigma)
\]

\[
= \bar{\rho}
\]

We can compute the polarization vector by taking the trace of \( \rho \) with \( \sigma \)
This can be generalized to experiments with $N$ outcomes:

$$ S_n = I + \text{basis of } N \times N \text{ traceless Hermitian matrices} $$

$N$ isos $N^2-N$ complex off diagonal

there are $N^2$ independent Hermitian matrices

$$ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \ldots, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} $$

$$ \text{Tr } I = N $$

$$ S_n = \frac{1}{N} \text{Tr}(S_n) I \quad \text{is traceless and independent of } I $$

we can find $N^2-1$ traceless Hermitian matrices:

$$ \frac{1}{N} \text{Tr}(S_n^T S_m) = \langle S_n | S_m \rangle $$

Using Gram Schmidt method
we can define linear combinations that are orthonormal and traceless (since linear combinations of traceless matrices are traceless)

$$ \text{Tr}(A+\alpha B) = \text{Tr}(A) + \alpha \text{Tr}(B) $$
These can be normalized so

\[ \text{Tr } S_1 = N \]
\[ \text{Tr } S_n = 0 \quad n \neq 1 \]
\[ \text{Tr } (S^+_n S_m) = \text{Tr } (S_n S_m) = \delta_{nm} N \]

Since these are a basis for Hermitian matrices,

\[ \rho = \sum p_n S_n \]
\[ \text{Tr } \rho = 1 = \sum p_n \text{Tr } S_n = p_1 N \]
\[ p_1 = \frac{1}{N} \]
\[ 1 \geq \text{Tr } \rho^2 = \sum p_n p_m \text{Tr } (S_n S_m) = \sum p_n^2 \delta_{nm} N \]
\[ = \frac{N}{N^2} + \sum_{n \neq 1} \frac{1}{N} p_n^2 N \]
\[ N \sum_{n=1} \frac{p_n^2}{N} \leq 1 - \frac{1}{N} \]
\[ \leq \sum p_n^2 \leq \frac{1}{N} - \frac{1}{N^2} = \frac{N - 1}{N^2} \]

(when \( |p_n| = \frac{1}{N^{\frac{1}{2}} N^{-\frac{1}{2}}} = 1 \) pure state)
quantum entropy

\[ S = -k \, \text{Tr} \left( \rho \ln \rho \right) \]

\( k = \text{Boltzmann constant} \)

If \( \rho \) is a pure state \( \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = \rho \)

\[ S = -k \, 1 \cdot \ln 1 = 0 \]

If \( \rho \) has equal prob for all state

\[ S = -k \sum_i \frac{1}{N} \ln \frac{1}{N} \]

\[ = -k \frac{N}{N} \ln \frac{1}{N} \]

\[ = -k \ln N \]

\[ = k \ln N \]

so the density matrix with \( \frac{1}{N} \)

on the diagonal has max entropy than the pure state.

To find the state with largest entropy consider

\[ S = 0 \text{ subject to } \text{Tr} \rho = 1 \]
using lagrange multipliers

\[ s \left( -k \sum p_n \ln p_n - x \left( \sum p_n - 1 \right) \right) = 0 \]

\[ \left( -k \sum p_n \ln p_n - k \sum p_n \frac{1}{p_n} \frac{1}{p_n} - 1 \sum 2 p_n \right) = 0 \]

so this to be extremal \( \forall n \) for every \( p_n \)

\[-k \ln p_n - k - x = 0 \]

\[-k \ln p_n = k + x \]

\[\ln p_n = -1 - \frac{x}{k} \]

\[p_n = e^{-1 - \frac{x}{k}} \]

\[\sum p_n = 1 = N e^{-1 - \frac{x}{k}} \]

\[e^{-1 - \frac{x}{k}} = \frac{1}{N} = \rho_n \]

so the state of maximal entropy has \( p_n = \frac{1}{N} \) \( \forall n \) in all \( N \).
For a given density matrix the mean energy \[
\langle E \rangle = \text{Tr} (\rho H)
\]
where \( H \) is the Hamiltonian. In equilibrium
\[
\frac{d\rho}{dt} = -\frac{i}{\hbar} [H, \rho] = 0
\]
so in equilibrium \( H \) commutes with \( \rho \)

\text{Note:}
\[
\frac{d\rho}{dt} = \frac{d}{dt} \sum_{n} |a_n\rangle \langle a_n| \rho |a_n\rangle <a_n| = \frac{\partial}{\partial t} |a_n\rangle \langle a_n| \rho |a_n\rangle <a_n| + |a_n\rangle \langle a_n| \frac{\partial}{\partial t} \rho |a_n\rangle <a_n| = \frac{\hbar}{i} [H, |a_n\rangle \langle a_n|] = \hbar |a_n\rangle \langle a_n| H
\]
this is opposite to the sign in the Heisenberg equations
In equilibrium, \( H, p \) can be simultaneously diagonalized

\[
\text{Tr} (H \rho) = \sum \text{Tr} (a_n \rho) = \langle a_n \rangle = \langle E \rangle
\]

\[
\text{Tr} (\rho) = 1
\]

We can try to maximize the entropy subject to both constraints. This requires adding 1 more Lagrange multiplier.

\[
S (\text{Tr} (-k \ln \rho - \lambda \rho - \gamma \rho H)) = 0
\]

\[
\sum (-k \delta \rho_n \ln \rho_n - k \rho_n \rho_n \delta \rho_n - \lambda \delta \rho_n - \gamma \delta \rho_n H_n) = 0
\]

\[
-\ln \rho_n - k - \lambda - \gamma H_n = 0
\]

\[
\ln \rho_n = \frac{1}{k} (-\lambda - \gamma H_n)
\]

\[
\rho_n = e^{-\frac{1}{k} \left( -\lambda - \gamma H_n \right)}
\]

\[
\sum \rho_n = 1
\]

\[
\rho_n = \frac{e^{\frac{1}{k} \left( -\lambda - \gamma H_n \right)}}{\sum e^{\frac{-\lambda - \gamma H_n}{k}}} = \frac{e^{-\frac{\gamma H_n}{k}}}{\sum e^{-\frac{\gamma H_n}{k}}}
\]
where $\gamma$ is determined by

$$
\sum e^{-\frac{\gamma E_n}{kT}} E_n = \langle E \rangle
$$

$$
Z = \sum e^{-\frac{\gamma E_n}{kT}}
$$

$$
\frac{\partial \ln Z}{\partial T} = \frac{1}{Z} \sum e^{-\frac{\gamma E_n}{kT}} (-\frac{E_n}{kT})
$$

$$
\langle E \rangle = -k \frac{\partial \ln Z}{\partial T}
$$

We can identify $\gamma = \frac{1}{T}$ where $T$ is the temperature and $Z$ is the partition function.
Symmetries and conservation laws

Last semester

\[ U(\theta) = e^{-\frac{i}{\hbar} \hat{H} \theta} \]

was the unitary operator that implemented rotations. If \( H \) is a Hamiltonian satisfying

\[ H = U(\theta) H U^+(\theta) \]

then it is invariant with respect to rotations about the \( \hat{\theta} \) axis.

\[ \frac{d}{d\theta} H = 0 = -\frac{i}{\hbar} U(\theta) (\hat{\theta} \cdot \hat{H} - H \hat{\theta}) U^+(\theta) \]

since \( U(\theta) \) is unitary it has an inverse which gives

\[ [\hat{\theta}, H] = 0 \]

These are equivalent statements since

\[ U(\theta) H U^+(\theta) = H + \sum_{n=1}^{\infty} \left( -\frac{i}{\hbar} \right)^n \frac{1}{n!} [\hat{\theta}, [\hat{\theta}, \cdots [\hat{\theta}, H] \cdots ] = 0 \]

\[ \hat{\theta} \]
If $[\hat{J}, H] = 0$ then the Heisenberg equations of motion imply

$$\frac{d\hat{J}}{dt} = \frac{i}{\hbar} [H, \hat{J}] = 0$$

which means that $\hat{J}$ is a conserved quantity in time. In this case the conserved quantity is the projection of the angular momentum on the $\hat{n}$ axis.

The condition $[H, \hat{J}_z] = 0$ also means that it is possible to find simultaneous eigenstates of $H$ and $\hat{J}_z$.

$$(\hat{J}_z) |E\psi\rangle = \lambda |E\psi\rangle$$

$H |E\psi\rangle = E |E\psi\rangle$

If $[H, \hat{J}_z] = 0$ for any $\hat{n}$ while $|E\psi\rangle$ is not an eigenstate of $\hat{J}_z$ ($\hat{n} \neq \hat{n}$).

$$H \hat{J}_z |E\psi\rangle = (\hat{J}_z) H |E\psi\rangle$$
$$= H (\hat{J}_z) |E\psi\rangle$$

which means $(\hat{J}_z) |E\psi\rangle$ is an eigenstate of $H$ with energy $E$. 

In general it will be a linear combination of \( |E \lambda \rangle \) for different values of \( \lambda \)

\[
| \tilde{\phi}_i \rangle |E \lambda \rangle = \sum |E_n \rangle
\]

The coefficients can be found using raising and lowering operators.

\[
H = U(\theta \tilde{\phi}_i) H U^+(\theta \tilde{\phi}_i)
\]

is an example of a symmetry.

\[
[H, \tilde{\phi}_i]\n\]

is a conservation law that comes from the symmetry.

1 parameter unitary groups

\[
U(\theta \tilde{\phi}_i) U(\theta_2 \tilde{\phi}_i) = U(\theta_1 + \theta_2 \tilde{\phi}_i)
\]

\[
U(0, \tilde{\phi}_i) = I
\]

\[
U^+(\theta \tilde{\phi}_i) = U(-\theta, \tilde{\phi}_i)
\]

These three conditions define what we mean by a unitary 1 parameter group.
abstractly

\[ \mathbf{U}(\lambda_1, \lambda_2) = \mathbf{U}(\lambda_1 + \lambda_2) \]
\[ \mathbf{U}^+(\lambda_1) = \mathbf{U}(-\lambda_1) \]
\[ \mathbf{U}(\Omega) = \mathbf{I} \]

consider

\[ \mathbf{I} = \mathbf{U}(\lambda_1) \mathbf{U}(\lambda_2) \mathbf{U}^+(\lambda_2) \mathbf{U}^+(\lambda_1) \]
\[ \mathbf{0} = \frac{d\mathbf{U}}{d\lambda}(\lambda_1) \mathbf{U}(\lambda_2) \mathbf{U}^+(\lambda_2) \mathbf{U}^+(\lambda_1) + \]
\[ \mathbf{U}(\lambda_1) \mathbf{U}(\lambda_2) \mathbf{U}^+(\lambda_2) \mathbf{U}^+(\lambda_1) \frac{d\mathbf{U}^+}{d\lambda}(\lambda_1) \]
\[ \left( \frac{d\mathbf{U}}{d\lambda}(\lambda_1) \mathbf{U}(\lambda_2) \mathbf{U}^+(\lambda_2) \mathbf{U}^+(\lambda_1) \right)^\dagger \]

\[ \therefore \quad \frac{d\mathbf{U}}{d\lambda}(\lambda_1) \mathbf{U}(\lambda_2) \mathbf{U}^+(\lambda_2) \mathbf{U}^+(\lambda_1) = \]
\[ - \left( \frac{d\mathbf{U}}{d\lambda}(\lambda_1) \mathbf{U}(\lambda_2) \mathbf{U}^+(\lambda_2) \mathbf{U}^+(\lambda_1) \right)^\dagger \]

since \( \mathbf{U}(\lambda_2) \mathbf{U}^+(\lambda_2) = \mathbf{I} \) this means

\[ \mathbf{F}(\lambda_1) = \frac{d\mathbf{U}}{d\lambda}(\lambda_1) \mathbf{U}^+(\lambda_1) = - \mathbf{F}^+(\lambda_1) \]
\[ \mathbf{F}(\lambda_1 + \lambda_2) = \frac{d}{d\lambda} \mathbf{U}(\lambda_1 + \lambda_2) \mathbf{U}^+(\lambda_1 + \lambda_2) \]
\[ = \frac{d}{d\lambda_1} \mathbf{U}(\lambda_1 + \lambda_2) \mathbf{U}^+(\lambda_1 + \lambda_2) \]
\[ = \frac{d\mathbf{U}}{d\lambda_1} \mathbf{U}^+(\lambda_1) = \mathbf{F}(\lambda_1) \]
These equations, which follow from the conditions:

\[ F^t = -F, \quad F \text{ is independent of } t \]

\[ \frac{du}{dt} U^t(t) = F = it \]

\[ \frac{du}{dt} = i\beta U(t) \]

\[ \frac{d^nu}{dt^n} = (i\beta)^n U(t) \]

\[ U(t) = \sum_{n=0}^{\infty} \frac{1}{n!} (i\beta)^n U(0) = e^{it\beta} \]

so in general one parameter unitary groups have the form

\[ e^{it\beta} \quad \beta = \beta^* \quad \frac{d\beta}{dt} = 0 \]

Consider

\[ U(\alpha) \langle x|\psi \rangle = \langle x-\alpha|\psi \rangle \]

This operator moves the center of the wave function to the right by \( \alpha \),

\[ U(\alpha_1)U(\alpha_2) \langle x|\psi \rangle = U(\alpha_1) \langle x-a_2|\psi \rangle \]

\[ = \langle x-a_1-a_2|\psi \rangle = U(a_1+a_2) \langle x|\psi \rangle \]
\[ \langle x | \psi \rangle = \frac{1}{\sqrt{2}} (\langle x | 0 \rangle + \langle x | 1 \rangle) \]

\[ \int \langle x | \psi \rangle \langle \psi | x \rangle \, dx = \end{align*} \]

\[ \phi = x + a \]

\[ \int \langle \psi | x - a \rangle \langle x - a | \psi \rangle \, dx \]

\[ \int \langle \psi | x \rangle \langle x | U^\dagger(a) U(a) \psi \rangle \]

This means

\[ \langle \psi | \phi \rangle = \langle \psi | U^\dagger(a) U(a) | \phi \rangle \]

For any \( |\psi \rangle \) and \( |\phi \rangle \), so \( U^\dagger(a) U(a) = I \). This means that translations in one direction is a one parameter unitary group.

\[ i \frac{d}{da} \langle x | \psi \rangle = \frac{d}{da} \left( \langle U(a) \psi | \psi \rangle \right) \]

\[ = \frac{d}{da} \langle x - a | \psi \rangle \big|_{a=0} \]

\[ = -i \frac{d}{dx} \langle x | \psi \rangle \]

\[ \xi = i \frac{d}{dx} = -\frac{i}{\hbar} \mathbf{p} \quad \mathbf{p} = \frac{\hbar}{i} \frac{d}{dx} \]

\[ U(a) = e^{-i \mathbf{p} \cdot \mathbf{a} / \hbar} \]

In this case, \( \mathbf{p} \) is the momentum operator.
If \( [H, U(t)] = 0 \) then \( [H, \vec{p}] = 0 \).

This means that if \( H \) is invariant under translations then linear momentum is conserved.

The Heisenberg equations of motion imply:

\[
\frac{dH}{dt} = \frac{i}{\hbar} [H, \hat{H}] + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}
\]

If \( H \) has no explicit time dependence then \( H = \) energy is conserved. Since

\[ i\hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle \]

and \( H \) is independent of time

\[ U(t) = e^{-\frac{i}{\hbar} H t} \]

The appearance of \( \hbar \) is to make \( \hat{p}, \hat{\phi}, \hat{\hat{p}} \) dimensionless. They all have units of momentum \( \times \) displacement.
diversion on measurement

In a quantum theory we have encountered 3 kinds of measurement:

\[ |\Psi\rangle |^2 = \text{probabilities} \]

\[ \langle \Psi | A | \Psi \rangle = \sum K_{\alpha \beta} \langle \alpha | A | \beta \rangle \langle \beta | \alpha \rangle = \text{expectation values} \]

\[ \text{Tr}(\rho A) = \sum K_{\alpha \beta} \langle \alpha | A | \beta \rangle \langle \beta | \alpha \rangle = \text{ensemble average} \]

under a unitary transformation:

\[ |\Psi\rangle \rightarrow |\Psi'\rangle = U |\Psi\rangle \]

\[ A \rightarrow A' = U A U^+ \]

\[ A' |\lambda_{\alpha} \rangle = U A U^+ |\lambda_{\alpha} \rangle = \lambda_{\alpha} U |\lambda_{\alpha} \rangle \]

Eigenvalues of \( A \) do not change.

Classical probabilities are numbers so they are not affected by unitary transformations.
Galilean Relativity

In classical systems there are inertial coordinate systems that preserve the form of Newton's second law for a free particle

\[ m \frac{d^2 \vec{x}}{dt^2} = 0 \]

If we let

\[ \vec{x}' = R \vec{x} + \vec{v} t + \vec{a} \]

\[ t' = t + t_0 \]

\[ \frac{d}{dt'} = \frac{dt}{dt'} \frac{d}{dt} = \frac{d}{dt} \]

\[ \frac{d^2 \vec{x}'}{dt'^2} = \frac{d^2 \vec{x}}{dt^2} = R \frac{d^2 \vec{x}}{dt^2} + \vec{v} \frac{d^2 t}{dt^2} + \frac{d^3 \vec{a}}{dt^3} \]

\[ m \frac{d^2 \vec{x}'}{dt'^2} = R(m \frac{d^2 \vec{x}}{dt^2}) = 0 \]

where \( R \) is a rotation matrix, since \( R \) has an inverse this means

\[ m \frac{d^2 \vec{x}'}{dt'^2} = m \frac{d^2 \vec{x}}{dt^2} = 0 \]
If we require the laws of physics to have the same form in these coordinate systems then

\[ F_i(x, x_0, v_i, \bar{v}_0, t) \]

1. is independent of time
2. is a function of coordinate differences
3. is a function of velocity differences
4. is rotationally covariant

\[ F(Rx_1, \ldots, Rx_n, Rv_1, \ldots, Rv_n) = F(x_1, \ldots, x_n, v_1, \ldots, v_n) \]

\[ \begin{align*}
\bar{x}'' &= R_2 x' + v_2 t' + a_2 \\
n' &= t' + t_0
\end{align*} \]

\[ \begin{align*}
x' &= R_1 \bar{x} + v_1 t + a_1 \\
n' &= t + t_0
\end{align*} \]

\[ \begin{align*}
\bar{x}'' &= R_2(R_1 \bar{x} + v_1 t + a_1) + \bar{v}_2(t - t_0) + \bar{a}_2 \\
n'' &= t + t_0 + t_0'
\end{align*} \]
\[
\begin{align*}
\vec{x}'' &= (R_2 R_1) \vec{x} + (R_2 \vec{v}_1 + \vec{v}_2) t + R_2 \vec{a}_1 + \vec{v}_2 t_0 + \vec{a}_2 \\
&= R_{21} \vec{x} + \vec{v}_2, t + \vec{a}_2 \\
\end{align*}
\]

This shows that successive Galilean transformations result in a new Galilean transformation:

\[
R_2 = R_1, \quad \vec{v}_2 = -R_1 \vec{v}_1, \quad \vec{a}_2 = -R_1 \vec{a}_1 + R_1 \vec{v}_1 t_0, \quad t_2 = t_1 - t_0
\]

give

\[
\begin{align*}
\vec{x}'' &= \vec{x}, \quad t'' = t
\end{align*}
\]

so each Galilean transformation has an inverse, the identity has

\[
R = I, \quad \vec{v} = 0, \quad \vec{a} = 0, \quad t_0 = 0
\]

these transformations can be represented in matrix form:

\[
\begin{pmatrix}
\vec{x}'' \\
\vec{t}''
\end{pmatrix} =
\begin{pmatrix}
R & \vec{v} & \vec{a} \\
0 & 1 & t_0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\vec{x}' \\
\vec{t}'
\end{pmatrix} =
\begin{pmatrix}
R \vec{x} + \vec{v} t + \vec{a} \\
\vec{t} + t_0
\end{pmatrix}
These transformations form a group

\[ g(R, \gamma, a, \tau) \]

\[ \times \quad g_1, g_2 = g_3 \quad \text{is an element of the group} \]

\[ g_1 \rightarrow g_1^{-1}, g_1 = I \quad \text{every element has an inverse} \]

\[ g(I_{000}) = I \quad \text{the identity is an element of the group} \]

\[ g_1(q_1, q_3) = (q_1, q_2)g_3, \quad \text{matrix multiplication is associative} \]

**Wigner's Theorem**

A correspondence \( |\psi\rangle \rightarrow |\psi'\rangle \)

that preserves quantum probabilities can only be implemented by a unitary or antunitary transformation

\[ \times \text{unitary} \quad \langle \psi' | \phi' \rangle = \langle \psi | \phi \rangle \]

\[ \times \text{antunitary} \quad \langle \psi' | \phi' \rangle = \langle \phi | \psi \rangle \]

In both cases

\[ \langle \psi' | \phi' \rangle^2 = \langle \psi' | \phi' \times \phi' | \psi' \rangle = \langle \phi | \psi \rangle \langle \psi | \phi \rangle = \langle \psi | \phi \rangle^2 \]
so it is clear that both types of transformations preserve quantum probabilities. The part of Wigner's theorem that is not true is the converse.

The proof of the converse can be found in Jost's quantum mechanics book.

Anti-unitary transformations are anti-linear, which means

\[ A (\alpha |\psi_1\rangle + \beta |\psi_2\rangle) = \alpha^* (A|\psi_1\rangle) + (A|\psi_2\rangle) \]

Note

\[ \langle \psi_1' | \psi_1 + \alpha \psi_2' \rangle = \langle \phi_1 + \alpha \phi_2 | \psi_1 \rangle \]

\[ = \alpha \langle \phi_1 | \psi_1 \rangle + \alpha^* \langle \phi_2 | \psi_1 \rangle \]

\[ = \langle \psi_1' | \phi_1 \rangle + \alpha^* \langle \psi_1' | \phi_2 \rangle \]

since this must hold for all \( 1 \psi_1 \rangle \)

\[ (\phi_1' + \alpha \phi_2' \rangle = 1 \phi_1' \rangle + \alpha^* 1 \phi_2' \rangle \]
If $H$ is Hermitian it has real eigenvalues. If $W$ is antiunitary, and $WHW^{-1} = H$

then

$$WH |\psi\rangle = W |\lambda \psi\rangle = \lambda^* W |\psi\rangle$$

$$= \lambda W |\psi\rangle$$

$$(WHW^{-1}) |\psi\rangle = \lambda W |\psi\rangle$$

$$(H \chi) W |\psi\rangle = \lambda W |\psi\rangle$$

antiunitary transformations preserve the eigenvalues of Hermitian operators.

If unitary and antiunitary operators preserve all quantum observables

If quantum measurement cannot be used to distinguish inertial coordinate systems then by Wigner's theorem changes in inertial coordinate systems are given by unitary transformation $U(q)$
where \( y \) represents a Galilean transformation. Since \( q_3 = q_2 q_1 \)

\[
U(q_3) = U(q_2) U(q_1)
\]

Remark - If \( U(\lambda) U(\lambda') = U(\lambda \lambda') \) then \( U(\lambda) = U(\lambda/2) U(\lambda/2) \) so if \( U(\lambda) \) is antiunitary \( U(\lambda) \) is unitary - but \( \lambda \) is arbitrary one parameter groups of observable preserving transformations must be unitary.

\[
U(q_3) |\psi\rangle = |\psi'\rangle
\]

\[
U(q_2) U(q_1) |\psi\rangle = |\psi''\rangle
\]

Since these transformations are the same at most they can differ by a phase

\[
|\psi''\rangle = e^{i \phi(q_2, q_1)} |\psi'\rangle
\]

\[
U(q_1) U(q_2) = e^{i \phi(q_1, q_2)} U(q_1 q_2)
\]
Thus
\[ g_v(c) g_{\epsilon}(t_0) g_{\nu}(v) = g_{\nu}(v) g_{\epsilon}(a-\nu t_0) g(t_0) \]
apply this to \( |E\epsilon\rangle \) (simultaneous eigenstate of energy and momentum

\[ U(g_v(c) g_{\epsilon}(t_0)) |E\epsilon\rangle = \]

\[ U(g_v(c) g_{\epsilon}(t_0)) U(g_{\nu}(v)) |E\epsilon\rangle = \]

\[ U(g_{\nu}) U(g_{\epsilon}(a-\nu t_0)) |E\epsilon\rangle = \]

\[ U(g_{\nu}) e^{-i\frac{p_a(a-\nu t_0)}{2m} - iE_\epsilon t_0} g_{\nu}(v)|E\epsilon\rangle = \]

\[ U(g_{\nu}) e^{-i\frac{(E-\nu t_0)}{2m} - i\frac{p_a}{2}} g_{\nu}(v)|E\epsilon\rangle \]

This means
\[ U(g_{\epsilon}(v)) |E\epsilon\rangle \]
is an eigenstate of \( H_{\tilde{\epsilon}} \) with eigenvalues \( E - \nu \epsilon, \epsilon \)

On the other hand what we expect is
\[ p \rightarrow p' - m\tilde{v} \]
\[ E = \frac{p^2}{2m} \rightarrow (\frac{p - m\tilde{v}}{2m})^2 = \frac{p^2}{2m} - \nu \epsilon + \frac{m\tilde{v}^2}{2} \]
the transformation is missing the \( m\tilde{v} \) and \( \frac{m\tilde{v}^2}{2} \) terms.
This is called a unitary ray representation of the Galilean group.

In some cases it is possible to redefine \( U(q_1) \rightarrow U(q_1) e^{i\phi(q_1)} \)

\[
U(q_1) U(q_2) = e^{i(\phi(q_1,q_2) - \phi(q_1) - \phi(q_2))} U(q_1,q_2)
\]

so the phases cancel. Sometimes this is not possible. It turns out that for Galilean transformations that change the velocity of the rest frame the phases cannot be transformed away, to see this first define

\[
g(\bar{a}) \; \; ; \; x \rightarrow x + \bar{a} \\
g_v(v) \; \; ; \; x \rightarrow x + vt \\
g_R(R) \; \; ; \; x \rightarrow Rx \\
g_t(t_o) \; \; ; \; t \rightarrow t+t_o
\]

Consider

\[
g(a) g(t_o) g(v) \; \; ; \; x \rightarrow x + vt \begin{cases} x+vt & t+to \\ x+vt+a & t+t_0 \\ x+a & t+t_0 \\ x+a+vt & t+t_0 \\ x+a - vt & t+t_0 \end{cases}
\]

\[
g(v) g(a-vt_o) g(t)
\]
**Remarks** - the correct quantity involves in which has not been previously defined.

Ignoring the possibility of missing phases, the translation is wrong.

Bergmann, Ann Math 59, 1, 1954, showed that the correct result could be recovered using:

\[
\mathcal{U}(q_1) \mathcal{U}(q_2) = \mathcal{U}(q_1, q_2) \in \text{i} \mathcal{Q}(q, q_2)
\]

\[
\Phi(q_1, q_2) = \frac{m}{2} (\dot{q}_1 \dot{q}_2 \ddot{q}_2 - \ddot{q}_1 \dot{q}_2 \dot{q}_2^2 + \nu_1 \nu_2 \dot{q}_2 + \nu_1 \nu_2 \dot{q}_1)
\]

Using this phase recovers the correct translation.