Lecture 31

Relation between scattering operator and potential

Assume

1. \( AA^+ = I \)
2. \( \lim_{t \to \pm \infty} \| (I - A) \psi \| = 0 \)

for both time limits

For

\( H' = A^+ HA \)

1. \( \lim_{t \to \pm \infty} e^{i H' t / \hbar} = A \Omega^\pm \)

\( S^1 = \Omega_+^\dagger \Omega_- = \Omega_+ AA^+ \Omega_- = \Omega_+ \Omega_- = S \)
\textbf{Proof}

\[ \lim_{t \to \pm \alpha} \| (e^{iHt/n} - iH_\pm t/n) e^{-iHt/n} \| \times |\psi> \|
\]

\[ = \lim_{t \to \pm \alpha} \| A^+ (e^{iH_\pm t/n} \psi - e^{-iH_\pm t/n} \psi) \| \times |\psi> \|
\]

\[ = \lim_{t \to \pm \alpha} \| A^+ e^{iH_\pm t/n} (A - I) e^{-iH_\pm t/n} |\psi> \|
\]

\[ = \lim_{t \to \pm \alpha} \| (A - I) e^{-iH_\pm t/n} |\psi> \| = 0
\]

\textit{x \ remarks}

\[ |\psi_\pm> = \Omega_\pm |\psi_\pm> \]

\[ = A^+ \Omega_\pm |\psi_\pm> \]

\[ = A^+ |\psi_\pm> \]

\textit{This means that while S does not change}
the scattering wave functions change — but because phase shifts are a property of $S = e^{2\pi i s}$, the phase shifts remain unchanged.

* we use the fact that

$$\lim_{t \to \pm \infty} ||(1-A) e^{-iHt/\hbar} 1+|| = 0$$

for both time limits in order to show

$\Omega'_+ = A\Omega_+$ and $\Omega'_- = A\Omega_-$

More interesting — this is a necessary and sufficient condition for $H$ and $H'$ to have the same $s$
to show the converse

(1) Assume \( S^1 = S \) and
\[
\begin{align*}
S^1 & = \Omega_+ \Omega_+ \quad S = \Omega_+ \Omega_-
\end{align*}
\]
\[
\Omega_+ \Omega_- = S^1 = S = \Omega_+ \Omega_-
\implies
\begin{align*}
\Omega_+ \Omega_+ = \Omega_+ \Omega_+
\end{align*}
\]

(2) Note that \( \Omega_+ |\varphi\rangle = 0 \) for bound states - define
\[
A^+ = \Omega_+ \Omega_+ + \frac{\hbar}{\sqrt{n}} b_n \times b_n^k
\]
then
\[
A^+ A = AA^+ = I
\]
\[
A^+ H = H' A^+ \quad \text{or}
\]
\[
H' = A^+ H A
\]
\[
\Omega^+ \Omega_+ = \Omega_+ \Omega_+ \Omega_+ \Omega_+ = \Omega_+
\]
\[
0 = \lim \| (A^+ e^{i H t / h} - e^{i H t / h} A^+) \psi \| \\
= \lim \| A^+ e^{i H t / h} (1 - A) e^{-i H t / h} \psi \| \\
= \lim \| (1 - A) e^{-i H t / h} \psi \| \\
\]

This shows that \( S = S' \) means that

\[ H' = A^+ H A \]

with

\[ \lim_{t \to \pm \infty} \| (1 - A) e^{-i H t / h} \psi \| = 0 \]

* This condition depends on \( A \) but not on \( V \).

* The fact that the wave function changes does not affect quantum observable.
In ordinary operators:

\[ 14' = A^+ 14 \]

\[ \sigma' = A^+ \sigma A \]

means ordinary observable and ensemble averages are unchanged.

Relativity and Quantum Mechanics

* Classically there are inertial coordinate systems where the laws of physics have the same form
In non-relativistic classical mechanics, different inertial coordinate systems are related by translations, rotations, time translation, and shifts by constant velocity.

In relativistic classical mechanics, the different inertial coordinate systems are related by translations, rotations, time shifts, and velocity changing Lorentz transformation.
Lorentz transformation preserve $c^2 \Delta t^2 - \Delta x \cdot \Delta x$ between events where $c$ is the speed of light in a vacuum.

* These have the property that the speed of light is the same in all inertial coordinate systems.

* Maxwell's equations are preserved under Lorentz transformations.

* Newton's Laws are preserved under Galilean transformations.
These are not consistent.

Michelson Morley experiment determined that different inertial coordinate systems are related by Lorentz transformations.

* Principle of relativity — experiments on isolated systems cannot distinguish different inertial coordinate systems classically — the equations of motion have the same form in all inertial coordinate systems.
In a quantum theory wave functions are not observable — Lorentz invariance in a quantum theory means that quantum measurements give identical results in all inertial coordinate systems. The observables are probabilities, expectation values and ensemble averages. These are left invariant by unitary transformations — Poincaré group.

Consider 2 space-time events $A$ and $B$ with coordinates $(t_A, \mathbf{x}_A)$ and $(t_B, \mathbf{x}_B)$.
define
\[ c^2 \Delta y_{ab} = c^2 (t_a - t_b)^2 - (x_a - x_b) \cdot (\bar{x}_a - \bar{x}_b) \]
where \( c \) is the speed of light in a vacuum. To give the space and time components identical units define
\[ x^0 = c \tau \]
we also define a metric tensor
\[ \eta_{ab} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \]
with this notation
\[ c^2 \Delta y_{ab}^2 = -\frac{1}{\eta_{\mu\nu}} \eta_{\mu\nu} (x_a - x_b)^\mu (x_a - x_b)^\nu \]
The Poincaré group is the group of transformations that leave $C^2 \Delta x^2$ invariant.

The most general transformation of the form

$$x^u \rightarrow x^u' = f^u(x)$$

that leaves $C^2 \Delta x^2$ invariant is

$$x^u \rightarrow x^u' = \sum \lambda^u \nu x^\nu + a^u$$

where $a^u$ is a constant 4 component vector and $\lambda^u$ is a constant matrix satisfying

$$\sum \lambda^u \alpha \lambda^v \beta \eta_{uv} = \eta_{\alpha \beta}$$
These transformations are called Poincare transforms. The result can be proved by using

\[\sum_{\mu} \eta_{\mu\nu} (x_A - x_B)^\mu (x_A - x_B)^\nu = \sum_{\nu'} \eta_{\nu'\nu} (x_A - x_B)_{\nu'} (x_A - x_B)_{\nu'}\]

and differentiating with respect to \(x_A^\mu, x_B^\nu\) and setting both to 0. (\(h\omega\))

We can write these as matrix equations:

\[x \rightarrow x' = \Delta x + a\]

\[\Delta = \Delta n A^T\]
The second equation is:

$$\det \eta = \det (\Lambda \eta \Lambda^T) = \det \Lambda \det \eta \det \Lambda^T = \det \Lambda \det \eta \det \Lambda$$

which gives

$$\det \Lambda^2 = 1$$

We also have

$$\eta_{\mu\nu} = \Sigma \Lambda_{\mu}^{\alpha} \Lambda_{\nu}^{\alpha} \eta_{\alpha\alpha}$$

$$= \Lambda_{\mu}^{\alpha} \Lambda_{\nu}^{\alpha} \eta_{\alpha\alpha} + 2 \Lambda_{\mu}^{\alpha} \Lambda_{\nu}^{\beta} \eta_{\alpha\beta} + \ldots$$

$$- 1 = (\Lambda_{\mu}^{\alpha})^2 (-1) + \Sigma (\Lambda_{\mu}^{\alpha})^2$$

$$\therefore (\Lambda_{\mu}^{\alpha})^2 = 1 + \Sigma (\Lambda_{\mu}^{\alpha})^2$$

There are 4 classes of Lorentz transformation
\[ \det \lambda = 1 \quad \lambda^2 \geq 1 \quad \Rightarrow \]

\[ \det \lambda = -1 \quad \lambda^2 \geq 1 \quad \Rightarrow \]

\[ \det \lambda = 1 \quad \lambda^2 \leq -1 \quad \Rightarrow \]

\[ \det \lambda = -1 \quad \lambda^2 \leq -1 \quad \Rightarrow \]

Class II includes space reflection.

Class II includes both space and time reflection.

Class IV includes time reflection.

The weak interaction is not invariant with respect to space reflection or time reversal.

The class of Lorentz transformations relevant for special relativity is class I \((\det \lambda = 1, \lambda^2 \geq 1)\)
These transformations satisfy

\[ x \rightarrow x' = \Lambda_1 x + a_1, \]
\[ x' \rightarrow x'' = \Lambda_2 (\Lambda_1 x + a_1) + a_2 \]

\[ = \Lambda_2 \Lambda_1 x + \Lambda_2 a_1 + a_2 \]
\[ = \Lambda_2 \Lambda_1 a_1 + \Lambda_2 a_2 \]

For \( x \xrightarrow{\lambda a} x' \) to preserve probabilities

\[ |\Psi'\rangle = U(\lambda a) |\Psi\rangle \]

For successive transformations

\[ x \xrightarrow{\Lambda_1 a_1} x' \xrightarrow{\Lambda_2 a_2} x'' \]

\[ U(\Lambda_2 a_2) U(\Lambda_1 a_1) = \Phi(12) \]

\[ U(\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2) \]

for the Poincare group it is possible to redefine phases so \( d(12) = 0 \)
Also while antiunitary transformations preserve probabilities, each transformation in it is a square, so the representation must be unitary.

Next consider

\[ X^\mu \delta_{\mu \nu} \rightarrow \left( \begin{array}{cc} x^0 + x^3 & x' - i x^1 \\ x^1 + i x^0 & x^0 - x^3 \end{array} \right) = X \]

Note

1) \( X = X^4 \)

2) \( \text{det} \ X = (x^0)^2 - \bar{x} \cdot x = c^2 s^2 \)

Any transformation that preserves \( X^4 = X \)
and \( \text{det} \ X \) is a Lorentz transformation.
\[ X' = AXA^+ \quad \text{det} \ A = 1 \]

\[ \text{det} \ X' = \text{det} (AXA^+) = \]
\[ = \text{det} (A) \text{det}(X) \text{det}(A^+) \]
\[ = 1 \cdot \text{det}(X) \cdot 1 \]
\[ (X')^+ = (AXA^+)^+ = AX^+A^+ = \]
\[ = AXA^+ = X' \]

**Using matrices**

\[ X' = AXA^+ + \sigma I \]

\text{Translation}

**Remark**

\[ \text{det} \ A = 1 \implies A = e \]
\[ \text{det} \ A = e \quad \tau-(c^6) = e \quad 2c \]

\[ C^0 = 0, \quad \frac{e}{2} \]
\[ A = \pm e \]
\[ \bar{z} = \text{complex vector} \]
spin 1 particle of mass \( m \)

\[
\rho^\mu = m \frac{d\gamma^\mu}{dt} = \frac{d^2\gamma^\mu}{dt^2}
\]

\[
c^2 = c^2 dt^2 - dx^2
\]

\[
c^2 = c^2 \left( \frac{dt}{ds} \right)^2 - \left( \frac{dx}{ds} \right)^2
\]

\[
c^2 = (\frac{dt}{ds})^2 (c^2 - V^2)
\]

\[
(\frac{dt}{ds})^2 = \frac{1}{1 - \frac{V}{c}} = \gamma^2
\]

\[
m \rho^\mu \rho^\nu = -m^2 \frac{c^2}{1 - \frac{V^2}{c^2}} + \frac{m^2 V^2}{1 - \frac{V}{c}}
\]

\[
= m^2 \left( \frac{V^2 - c^2}{1 - \frac{V}{c}} \right)
\]

\[
= -m^2 c^2
\]

\[
A = e^{-i \cdot \vec{E} \cdot \vec{r}} \quad \text{Rotation}
\]

\[
A = e^{i \cdot \vec{P} \cdot \vec{r}} \quad \text{Lorentz boost}
\]

\[
\cosh \frac{P}{2} = \frac{1 + \vec{P} \cdot \vec{r}}{2m c^2}
\]

\[
\sinh \rho = \frac{\vec{E}}{mc^2} \quad \sinh \rho = \frac{\vec{P}}{mc}
\]
Note
\[
A = (A A^\dagger)^{\frac{1}{2}} (A A^\dagger)^{-\frac{1}{2}}
\]
unitary (rotation)
positive (rotamless boost)
\[
= A (A^\dagger A) (A^\dagger A)^{-1}
\]

Particles - state of a particle of mass \( m \) and spin \( \frac{1}{2} \) is given by its momentum and magnetic quantum \( \pm \frac{1}{2} \).

Denote this state by
\[
| (m, \frac{1}{2}, \vec{p} u) \rangle
\]

\( \ast \) when \( \vec{p} = 0 \) and \( \lambda = R \) (rotation) \( U(R) \) can only change \( u \)
\[ U(R) |(m \varphi) \bar{\sigma} u \rangle = \]
\[ \frac{\sum}{V} l(m \varphi) \bar{\sigma} v \rangle <_{D\nu} U(R) |_{\bar{\sigma} u} \rangle = \]
\[ \frac{\sum}{V} l(m \varphi) \bar{\sigma} v \rangle D^{\nu}_{\bar{\sigma} u} (R) \]

\[ \star \]
\[ U(R) |(m \varphi) \bar{\sigma} u \rangle = \]
\[ \frac{\sum}{V} l(m \varphi) \bar{\sigma} v \rangle D^{\nu}_{\bar{\sigma} u} (R) \]

For \( \Lambda = B(P/m) = \gamma \)
\[ = \cosh(\frac{P}{2}) + \bar{P} \cdot \bar{\sigma} \sinh(\frac{P}{2}) \]

where
\[ \frac{E}{mc^2} = \cosh \rho \quad \frac{P}{mc} = \sinh \rho \]

Applying \( U(B(P/m)) \) to \( l(m \varphi) \bar{\sigma} u \rangle \) changes the momentum from 0 to \( \bar{P} \).
we define \( |(m \gamma) \bar{\psi} u \rangle \)
by
\[
| |(m \gamma) \bar{\psi} u \rangle = N \mathcal{U}(B(\theta/m)) |(m \gamma) \bar{\psi} u \rangle
\]

* The normalization constant is chosen to ensure \( \mathcal{U}(B(\theta/m)) \) is unitary.

* The interpretation of \( \bar{u} \) is that it is the spin that would be measured in the rest frame if we transform to the rest frame using \( B^{-1}(\theta/m) \).
For the choice of normalization
\[ \langle m g | \bar{\psi} u | \bar{m g} | \bar{\psi} u \rangle \]
\[ = \delta (\bar{p}' - \bar{p}) | \bar{\psi} u \rangle \]
\[ N = \sqrt{\omega m} \quad \omega m (\bar{p}) = \sqrt{\bar{p}^2 + m'^2} \]

\[ U(\beta (\bar{p}/m)) | (m g) \bar{\psi} u \rangle = \]
\[ | (m g) \bar{\psi} u \rangle \sqrt{\frac{\omega (\bar{p}/m)}{m c}} \]

Finally, for time evolution
\[ U(\tau, \sigma) | (m g) 0 u \rangle = \]
\[ e^{-i \sigma m g x} | (m g) 0 u \rangle \]

We can use the 3 red boxed equations to get \[ U(\lambda a) \] for a particle of mass \( m \) and spin \( J \).
\[ U(\Lambda a) | (m_2) \bar{\theta}u \rangle = \]
\[ U(Ia) U(\Lambda, 0) | (m_2) \bar{\theta}u \rangle \]
\[ U(Ia) U(\Lambda, 0) U(B(0), 0) | (m_2) \bar{\theta}u \rangle \]
\[ \times \sqrt{\frac{m}{\omega(r)}} = \]
\[ U(Ia) U(B(\Lambda 0/m)) U(B(\Lambda 0/m)) \times \]
\[ U(Ia) U(B(0), 0) | (m_2) \bar{\theta}u \rangle \]
\[ \sqrt{\frac{m}{\omega(r)}} = \]
\[ U(B(\Lambda 0/m)) U(I, \tilde{B}(\Lambda 0/m) a) \times \]
\[ U(\tilde{B}(\Lambda 0/m) \Lambda B(\Lambda 0/m)) | (m_2) \bar{\theta}u \rangle \]
\[ \sqrt{\frac{m}{\omega(r)}} = \]
\[ 0 \rightarrow \rho \rightarrow \Lambda \rho \rightarrow \rho \]
\[ U(B(\Lambda 0/m)) U(I, \tilde{B}(\Lambda 0/m) a) | (m_2) \bar{\theta}u \rangle \]
\[ D_{\rho u}^{\bar{\theta}} (\tilde{B}(\Lambda 0/m) \Lambda B(\rho)) \sqrt{\frac{m}{\omega(r)}} = \]
\[ \frac{-i m [\tilde{B}(\Lambda 0/m) a]}{\lambda \rho a / h} \]
\[ e^{i \lambda \rho a / h} \]
\[ U(\lambda a) \mid (m \gamma) \bar{p} n \rangle = \sum \mid (m \gamma) \Lambda_{\rho, \nu} \rangle \chi \sqrt{\frac{w(\lambda \rho)}{w(\lambda)}} \times \]
\[ D_{\gamma \mu}^{\lambda} (\mathbf{B}^{-1}(\lambda \rho) a \mathbf{B}^{(\theta)}) \times \]
\[ \left. \right| \Lambda_{\rho, \nu} \left. \right| \bar{p}, a / n \]

This gives an explicit unitary representation of the Poincaré group on the single particle Hilbert space.