Lecture 32

Special Relativity

* Inertial coordinate system: free particles move with constant velocity

* Experiment suggests different inertial coordinate systems are related by Poincare transformation:

\[(x^0, \vec{x}) = (ct, \vec{x})\]

\[\eta_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}\]

Poincare group = group of transformations
that preserve

\[ c^2 \mathcal{L}_{\eta} = \sum_{\mu \nu} \eta_{\mu \nu} (x_\mu - x_b) (x_\nu - x_b) \]

**Homework:** these have the form

\[
\begin{align*}
X^u &\rightarrow \bar{X}^u = \sum_{\alpha \beta} \lambda^u_{\alpha \beta} X^\alpha + a^u \\
\eta_{\alpha \beta} &\rightarrow \bar{\eta}_{\alpha \beta} = \sum_{\alpha \beta} \lambda^\alpha \lambda^\beta \eta_{\alpha \beta}
\end{align*}
\]

where \( \lambda^u_{\alpha \beta} \) and \( a^u \) are constants

\( \lambda^u_{\alpha \beta} \) is called a **Lorentz transformation**

Lorentz transformations can be classified by

\[
\det \lambda = \pm 1
\]

\( \lambda \geq 1 \), \( \lambda \leq -1 \)
The transformations relevant to special relativity are the transformations satisfying

$$\det \lambda = 1, \; \lambda^a_a = 1$$

This set contains the identity. The other sets contain space and time reflections which do not leave the weak interaction invariant.

4 vectors are quantities like $x^a$ that transform under Lorentz transformations like

$$x^a \rightarrow x'^a = \gamma \Lambda^a_{\; b} x^b$$
Since \( y^2 c^2 = g_{\mu\nu} x^\mu x^\nu \)

is an invariant quantity,

\[
\frac{d x^\mu}{d s} \text{ and } \frac{d^2 x^\mu}{d s^2} \frac{d}{d s} (m x^\mu) = p^\mu
\]

(4 velocity, 4 acceleration, 4 moment)

are all 4 vectors

Note

\[
\frac{d x^\mu}{d s} = \left( \frac{d x^0}{d s}, \frac{d \vec{x}}{d s} \right) = \\
= \left( \frac{d x^0}{d s}, \frac{d x^0}{d s} \frac{d \vec{x}}{d x^0} \right) = \\
= \frac{d x^0}{d s} (1 - \frac{\vec{V}}{c})
\]

to find \( \frac{d x^0}{d s} \)

\[
\frac{c^2 \Delta s^2}{(\Delta x^0)^2} = \frac{(\Delta x^0)^2 - (\Delta x^1)^2}{(\Delta x^0)^2}
\]

Taking the limit \( (\Delta x^0) \to 0 \)

\[
(\frac{c}{d s})^2 = 1 - \frac{V^2}{c^2}
\]
\[
\frac{d\gamma}{dx^0} = \frac{1}{2} \sqrt{1 - \frac{v^2}{c^2}}
\]

\[
\frac{dx^0}{dt} = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

\[\text{mis means}\]

\[
\frac{dx^\mu}{dt} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} (c, \vec{v})
\]

\[
p^\mu = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} (mc, m\vec{v})
\]

The relativistic form of the Newtonian second law is

\[
\frac{dp^\mu}{ds} = \xi^\mu
\]

where \(\xi^\mu = (0, \vec{F})\) in the particle's rest frame and transforms like a 4-vector.
In the absence of forces

\[ \frac{dE}{d\tau} = 0 \Rightarrow \]

\( p^\mu \) is conserved.

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

\begin{align*}
p^0 &= \gamma mc \\
\vec{p} &= \gamma m \vec{v}
\end{align*}

\( p^0 \) is identified with the energy and \( \vec{p} \) is the conserved relativistic momentum \( \pm m \vec{v} \)

Principle of special relativity in quantum mechanics (Wigner 1939)
In an isolated system quantum observables cannot be used to distinguish different inertial reference frames.

This is different than the classical formulation which focuses on preserving the form of the equations in all inertial coordinate systems (as is the case with Maxwell's equations and the relativistic form of Newton's second law).
The quantities that can be measured in quantum mechanics are

(1) probabilities \( P_{\alpha\beta} = \mathcal{K} |\langle \alpha | \beta \rangle|^2 \)

(2) expectation values
\[ \langle a | B | a \rangle = \mathcal{S} \left| \langle a | \beta \rangle \right|^2 \]

(3) ensemble averages
\[ \text{Tr} (\rho B) = \sum_{\alpha} P_{\alpha} \langle \alpha | B | \alpha \rangle \]

The only transformations that preserve these are unitary or antiunitary transformations.

Consider \( \Lambda_{\alpha\beta} \)
\[ x_2 = \lambda_1 x_1 + a_1 \]
\[ x_3 = \lambda_2 x_2 + a_1 \]
\[ = \lambda_2 (\lambda_1 x_1 + a_1) + a_2 \]
\[ = \lambda_2 \lambda_1 x_1 + \lambda_2 a_1 + a_2 \]
\[ = \lambda_{12} x_1 + a_{12} \]

Comparing these

\[ \lambda_{12} = \lambda_2 \lambda_1 \]
\[ a_{12} = \lambda_2 a_1 + a_2 \]

This means

\[ U(\lambda_2 a_2) U(\lambda_1 a_1) |\Psi_i\rangle = \]
\[ U(\lambda_2 x_1, \lambda_2 a_1 + a_2) |\Psi_i\rangle \]

Remarks

* Each \((\lambda a)\) can be expressed as a product of square of elementary transformation.

\[ U(\lambda a) \text{ cannot be antiunitary} \]
* The states could be written by a phase

\[ \mathcal{U}(a_2, a_1) \mathcal{U}(a_1, a_1) = \mathcal{U}(a_2, a_1, a_1 + \alpha) \]

\[ \mathcal{E}^{(2,1)} \]

It turns out in the Poincare group the phases can be eliminated by redefining the phases of the individual \( \mathcal{U}(a_1) \) (up to the phases that arise in half integral spins in rotating by \( 2 \pi \))

A relativistic quantum theory is defined by a unitary representation of the Poincare group acting on the Hilbert space
$\mathfrak{u}(\lambda_2 a_2) \mathfrak{u}(\lambda_1 a_1) = \mathfrak{u}(\lambda_2 \lambda_1, \lambda_2 a_1 + a_2)$

$\mathfrak{u}^+ (\lambda_2 a_1) \mathfrak{u}(\lambda_2 a_1) = I$

**$sl(2 \mathbb{C})$** It is useful to represent 4 vectors by 2x2 Hermitian matrices

\[ X = \left( \begin{array}{cc} x^0 + x^3 & x' - ix^2 \\ x' + ix^2 & x^0 - x^3 \end{array} \right) = 2X^* \sigma_\mu \\
\]

where

\[ \sigma_\mu = (I, \bar{\sigma}) \]

and $\bar{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices. Note

\[ X = X^+ \]

\[ X^u = \frac{1}{2} \text{Tr} (\sigma_\mu \overline{X}) \]

\[ \det X = (X^0)^2 - \overline{X} \cdot \overline{X} = c^2 s^2 \]
This means that any linear transformation that preserves

\[ X = X^+ \quad \text{det} \, X \]

is a Lorentz transformation
\[ X' = AXA^+ \]

with \( \text{det} \, A = 1 \) satisfies these conditions since

\[ (X')^+ = (A^+)^+X^+A^+ = AXA^+ \]

\[ \text{det} \, X' = \text{det} (A^+) \text{det} \, X \text{ det} \, A^+ \]

\[ = 1 \cdot \text{det} \, X \cdot 1^* \]

\[ = \text{det} \, X \]

* while we could matting \( A \) by \( \psi \neq 0 \) and \( A^+ \) by \( 1/\psi \), this would not change the transformation.
comments

(1) The group of 2x2 complex matrices with \( \det A = 1 \) is called \( SL(2\mathbb{C}) \).

(2) \( A \) and \( -A \) correspond to the same Lorentz transformation.

(3) We can write \( A = e^B \)

\[ \det A = e^{\text{Tr} B} \]

a general matrix

\[ B = \sum b^n \sigma_n \]

where in the general case the \( b^n \) are complex

\[ \text{Tr} B = 2 \sum b^n \text{Tr}(\sigma_n) = 2b \]

\[ \det A = 1 = e^{2b} \]
This means \( b_0 = 0 \) and \( i \pi \).

It follows that a general \( SL(2c) \) matrix has the form

\[
A = \pm \vec{b} \cdot \vec{c}
\]

where \( \vec{b} \) is a complex 3 vector — properties

1. \( X^l = AXA^\dagger \Rightarrow \sum X^{lu} \delta_{uv} = \sum \lambda \delta_u A^\dagger X^v \)

\[
\frac{1}{2} \text{Tr} (\vec{\sigma} X^u \delta_u) = \frac{1}{2} \text{Tr} (\vec{\sigma} \lambda \delta_u A^\dagger X^v)
\]

\[
X^l \lambda^u = \frac{1}{2} \text{Tr} (\vec{\sigma} A \delta_u \lambda^v) X^v
\]

\[
\lambda^u \lambda^v = \frac{1}{2} \text{Tr} (\vec{\sigma} A \delta_u \lambda^v)
\]

2. if \( \vec{b} \) is real

\[
A = A^\dagger
\]

and
If \( A = e^{\frac{R}{i}} = A^* A \)
\( = RR = R^+R \geq 0 \)

in this case \( A \) is a positive Hermitian matrix

\[ e^{i \theta} \quad \text{and} \quad e^{-i \theta} \]
\[ A = e^{i \theta} \quad A = e^{-i \theta} \]
\[ A^* A = I \]

then \( A \) is unitary

\( A(\lambda) = e^{i \theta} \)

\[ \lambda \geq 0 \]

\[ A(\lambda) = A(\lambda/2) A(\lambda/2) \]

\[ A(\lambda) \rightarrow I \text{ continuously as } \lambda \rightarrow 0 \]

(this shows that any Lorentz transformation is a square.)
For Poincaré transformation

\[ x \rightarrow x' = \Lambda x \Lambda^* + B \]

where \( B \) is a constant Hermitian matrix.

Construction of \( \mathcal{U}(1)_G \) for a single particle

Consider a particle like an electron. In order to characterize the state of an electron we need to know its linear momentum and the projection of its spin on a given axis.
By changing frames
the 3 components of the 4 momentum can take on any value.

\[ p^\mu = \left( mc \frac{1}{\sqrt{1 - u_k^2}}, m\tilde{v} \frac{1}{\sqrt{1 - \tilde{u}_k^2}} \right) \]

(note while \( u_k < c \), \( \frac{1}{\sqrt{1 - u_k^2}} \) gets large as \( u_k \to c \))

Also the \( J_k \) can take on 2 values.

The Hilbert space for a single electron is defined

\[ \psi_\mu(p) \]

\[ \sum_{\mu = \pm 1} \int \psi_\mu(\tilde{p}) \psi_\mu(\tilde{p}) \, d^3 p < \infty \]

and we write these as, \( \langle \tilde{p} \mu \mid \chi \rangle \)

\[ 2 p^\mu p^\nu \eta_{\mu\nu} = \frac{m^2 u^2 - m^2 c^2}{1 - u^2/c^2} \]

\[ = m^2 c^2 - \frac{u^2 c^2}{c^2 - u^1} = -m^2 c^2 \]
up to a factor of $c$
this is the rest mass of the electron. We write the basic state as
$$| (m\frac{1}{2}) \bar{u} u \rangle$$
where $m, \frac{1}{2}$ are the fixed mass and spin of the electron

$$U(\theta, \phi) \rightarrow U(R, \theta)$$
(rotation)

Consider
$$U(R) | (m\frac{1}{2}) \bar{u} u \rangle$$
rotation (do not change $\bar{u}$

$$U(R) | (m\frac{1}{2}) \bar{u} u \rangle =$$
$$\sum_{n=0}^{\infty} | (m\frac{1}{2}) \bar{u} u \rangle M_{nm}(R)$$
\( M_{\nu \mu} (R) \) is a unitary 2x2 matrix representing a rotation in the basis of eigenstates of \( J_z \). We know that this is just \( M_{\nu \mu} (R) = D^{1/2}_{\nu \mu} (R) \)

\[
0 \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix} (m_{1/2}) \bar{\mu} \mu = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix} (\bar{\nu} \mu) \bar{\nu} \mu = D^{1/2}_{\nu \mu} (R)
\]

Note - because rotations by 2\pi changes the sign of \( D(R) \) it is useful to replace lorentz transforms\( labels \) by \( sl(2c) \) labels.
Homework

\[ \Lambda^{\mu \nu} = \frac{1}{2} tr \left( \sigma_{\mu} \sigma_{\nu} - \frac{p_{\mu}}{2} \delta_{\mu \nu} - \frac{p_{\nu}}{2} \delta_{\mu \nu} \right) \]

\[ = \begin{pmatrix}
\cosh \rho & 0 & 0 & \sinh \rho \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \cosh \rho
\end{pmatrix} \]

\[ \Lambda^{\mu \nu} \left( \begin{array}{c}
mc \\
o \\
o \\
o
\end{array} \right) = \left( \begin{array}{c}
mc \cosh \rho \\
o \\
o \\
mc \sinh \rho
\end{array} \right) \]

These transformations change the 4-velocity

\[ p^0 = mc \cosh \rho \]

\[ \vec{p} = mc \sinh \rho \hat{\vec{p}} \]

we call \( A = e^{\frac{\vec{p} \cdot \hat{\vec{p}}}{2}} \) a rotationless Lorentz boost
It transforms \((mc^0)\) to \(p^\mu = (\gamma mc, \gamma m \vec{v}) = (p^0, \vec{p})\)

Let \(B(\bar{p}/m) = e^{\frac{p_0}{m} \cdot \vec{b}}\)

We define \(U(B(\bar{p}/m))\)

\[
\begin{align*}
| (m + \frac{i}{2}) \vec{p} u \rangle & \equiv \\
U(B(\vec{e})) | (m + \frac{1}{2}) \vec{e} u \rangle & \equiv N(\vec{p})
\end{align*}
\]

* With this definition, the spin projection does not change

* The normalization constant must be chosen so \(U(B(\vec{0}))\) is unitary
To choose \( N \) note

\[
\Theta(p^0) \delta \left( m^2 c^2 + \vec{p}^2 - (p^0)^2 \right) \, d^4 p =
\Theta(p^0) \delta \left( m^2 c^2 + p'^2 - (p^{'0})^2 \right) \, d^4 p'
\]

\[
d^3 p \, \frac{\delta \left( p^0 - \sqrt{p^2 + m^2 c^2} \right)}{2 \sqrt{p^2 + m^2 c^2}} = \frac{\delta \left( p^0 - \sqrt{p'^2 + m^2 c^2} \right)}{2 \sqrt{p'^2 + m^2 c^2}} \, d^3 p'
\]

Integration over \( p, p' \)

\[
\frac{d^3 p}{2 \sqrt{p^2 + m^2 c^2}} = \frac{d^3 p'}{2 \sqrt{p'^2 + m^2 c^2}}
\]

Define \( \omega_m(p) = \sqrt{m^2 c^2 + p^2} \)

\[
\int \mid \Phi \mid^2 \, d^3 p < \nu | \lambda > = I = \langle \nu | U | \lambda > \langle \nu | U^* | \lambda >
\]

\[
\int \lambda \nu \rangle \left(N \, d^3 p \, N < \lambda \nu \right) \lambda \nu = p'
\]

\[
\int \mid p' \mid^2 \, d^3 p' < \nu | \lambda > = \frac{d^3 p}{d^3 p}' \, \frac{\omega_m(p')}{\omega_m(p)} \cdot \frac{\omega_m(p)}{\omega_m(p')}
\]

\[
N = \sqrt{\frac{d^3 p}{d^3 p} \cdot \frac{\omega(p')}{\omega(p)}}
\]

\[
\mathcal{U}(\lambda | 1 \Phi > = 1 \lambda > \sqrt{\frac{\omega(p')}{\omega(p)}}
\]
This means,
\[
| (m \frac{1}{2}) \bar{\phi} u > =
\]
\[
U(\beta(p/m)) | (m \frac{1}{2}) \phi u > \sqrt{\frac{m}{\omega_n (p)}}
\]

This ensures unitarity.

Finally, we consider translations of rest state
\[
U(H,a) | (m \frac{1}{2}) 0 u > =
\]
\[
\sum_{n} e^{ip \cdot a/n} | (m \frac{1}{2}) 0 u > =
\]
\[
\sum_{n} e^{imc \cdot a/n} | (m \frac{1}{2}) 0 u >
\]
Note: The definition

\[ u(B, 0) \Gamma(m_g, \bar{m}u) \]

\[ = (m_\gamma, \bar{m}u) \sqrt{\frac{\omega(o)}{m}} \]

means that \( u \) is the \( z \) component of the electron spin that would be measured in the electrons rest frame if we used \( B'(\bar{p}/m) \) to transform to the rest frame.

We can combine these three elementary transformations to construct a unitary representation of the Poincare group in the electron's Hilbert space.
\[ u(\lambda a) \mid (m \omega) \vert \bar{p} u \rangle = \]
\[ u(\tau a) u(1, 0) \mid (m \omega) \vert \bar{p} u \rangle = \]
\[ u(\tau a) u(1, 0) u(B(\frac{\lambda p}{m})) (m \omega) \chi_{\omega} \sqrt{\frac{m}{\omega (0)}} \]
\[ u(\tau a) u(B(\frac{\lambda p}{m}), 0) u(B(\frac{\lambda p}{m}) \chi_{\omega}) \times \]
\[ u(\lambda, \omega) u(B(\frac{\lambda p}{m}), 0) (m \omega) \chi_{\omega} \sqrt{\frac{m}{\omega (0)}} \]
\[ u(B(\frac{\lambda p}{m}), 0) u(\tau B(\frac{\lambda p}{m}) a) \times \]
\[ u(B(\frac{\lambda p}{m}), 0) (m \omega) \chi_{\omega} \sqrt{\frac{m}{\omega (0)}} \]
\[ u(B(\frac{\lambda p}{m}), 0) u(\tau, \bar{B}(\frac{\lambda p}{m}) \chi_{\omega}) \]
\[ \frac{1}{(m \omega) \chi_{\omega}} D_{\omega m} (\bar{B}(\frac{\lambda p}{m}) \chi m (\frac{\lambda p}{m})) \sqrt{\frac{m}{\omega (0)}} \]
\[ \text{rotation} \]
\[ 0 \rightarrow \phi \rightarrow \omega \rightarrow 0 \]
\[ \text{(called Wigner rotation)} \]
\[ u(B(\frac{\lambda p}{m}), \omega) \in \Sigma \chi_{\omega} \sqrt{\frac{m}{\omega (0)}} \]
\[ u(\mu, a) \mid (m_t^\pm) \bar{\vec{p}} u \rangle = \]
\[ \sum \epsilon (m_t^\pm) \bar{\vec{p}} u \rangle \times \]
\[ D_{\nu}^{\pm} \left( B \left( \frac{\Lambda_0}{m} \right) \right) \left( \frac{B(\mu)}{m} \right) \sqrt{\frac{\omega(\mu)}{\omega(m)}} \]

This is unitary since it involves products of unitary operators.
This generalizes to arbitrary spins

\[ U(\Lambda, v) |(m, \tilde{p}) \tilde{\rho} n \rangle = \]

\[ \sum_{\nu} e^{i \Lambda \cdot p \cdot \nu} |(m, \tilde{p}) \tilde{\rho} (\frac{\nu}{m}) \rangle \sqrt{\frac{\omega_m (m \nu)}{\omega_m (\nu)} \times D_{\nu \nu} \left( \tilde{B} \left( \frac{\nu}{m} \right) \times B \left( \frac{\nu}{m} \right) \right)} \]

If we take \( U(\Lambda, v) = U(\Xi, tc \omega) \)

\[ U(t) |(m, \tilde{p}) \tilde{\rho} n \rangle = \]

\[ -i \rho \cdot c \frac{t}{\hbar} e^{i \rho \cdot c \frac{t}{\hbar} |(m, \tilde{p}) \tilde{\rho} n \rangle} \]

where \( \rho \cdot c = \gamma m c^2 = mc^2 \sqrt{1 - \gamma^2 c^2} \]

which is the relativistic energy

\[ e^{-i E t / \hbar} \]
It follows that

\[ i\hbar \frac{\partial}{\partial t} \langle \psi \rangle \rho \mu \nu \mid U(t) \rangle_\nu \]

which is the ordinary Schrödinger equation

\[ E = \hbar \omega ; \quad m^2 c^2 = \left( p^\mu \right)^2 - \left( \vec{p} \right)^2 \]

\[ E = \sqrt{m^2 c^2 + \vec{p}^2} \]

\[ i\hbar \frac{\partial}{\partial t} \langle \psi \rangle \rho \mu \nu \mid U(t) \rangle_\nu \]

\[ \sqrt{m^2 c^2 + \vec{p}^2} \langle \psi \rangle \rho \mu \nu \mid U(t) \rangle_\nu \]

This is called the relativistic Schrödinger equation. It is a valid equation, but it gave the wrong magnetic moment using

\[ \vec{p} \rightarrow \vec{p} - \frac{e}{c} \vec{A} \]