Lecture 40

Identical particles

Last time

to get the correct probabilistic interpretation of quantum mechanics there must be
a 1-1 correspondence between
physical states and rays
in the Hilbert space

Let $T_{ij}$ be the operator
that interchanges particles
i and j. We showed

$T_{ij}^2 = I \quad T_{ij} = T_{ij}^+$

If i and j are identical
particles then the
correspondence requires
\[ T_{ij} |\Phi\rangle = |\Phi\rangle e^{i\phi} \]
\[ T^2_{ij} |\Phi\rangle = |\Phi\rangle = |\Phi\rangle e^{2i\phi} \]

This requires \( \phi = 0, \pi \) which means

\[ T_{ii} |\Phi\rangle = \pm |\Phi\rangle \]

We found a contradiction if on a given physical state some \( T_{ij} \) had +1 eigenvalues and others had -1 eigenvalues.

3. A state representing a system of identical particles must be either symmetric or antisymmetric with respect to interchanging identical particles.
Symmetrization postulate

States of half integer spin identical particles are antisymmetric (Fermions)

States of integer spin identical particles are symmetric (Bosons)

Permutation group

\[ i \neq j \quad \sigma(i) \neq \sigma(j) \]

\[ 1 \rightarrow \sigma(1) \]
\[ 2 \rightarrow \sigma(2) \]
\[ \vdots \]
\[ n \rightarrow \sigma(n) \]

Properties

1. \( N! \) permutations for \( N \) objects
   (particles)
(2) product

$$\sigma_2 \cdot g_1(i) = g_2(g_1(i))$$

(3) every permutation is the product of pairwise transpositions

(4) every permutation is a product of either an even or odd # of transpositions

(5) $|g_1| = 1$ even # transpositions

$|g_1| = -1$ odd # transpositions

(6) $g_i g_j \neq g_j g_i$

the product of permutations does not generally commute
7. The inverse of a permutation is the product of the transpositions in reverse order.

\[ |\sigma| = 16 \] \( \Rightarrow \) \( 16! = 16^{-1} \)

8. \[ \frac{\sum_{\sigma} f(\sigma)}{6} = \sum_{\sigma} f(\sigma(6)|6) \]

9. Let \( e_i = \) even permutation

pick \( e_i \Rightarrow \) \( e_1, e_2 \) are odd permutation \( \Rightarrow \)

\[ \frac{N_1}{2} \] even permutation

\[ \frac{N_1}{2} \] odd permutation
Projects on symmetric or antisymmetric subspace of N particle Hilbert space

\[ P_\sigma = \text{permutation operator} \]
\[ \rho(\sigma) = \text{set of permutations on } N \text{ objects} \]

\[ S = \frac{1}{N!} \sum_{\sigma \in \rho(\sigma)} P_\sigma \]
\[ A = \frac{1}{N!} \sum_{\sigma \in \rho(\sigma)} (-1)^{s_{\sigma}} P_\sigma \]

*Note*

\[ S^2 = \frac{1}{(N!)^2} \sum_{\sigma, \sigma' \in \rho(\sigma)} P_\sigma P_{\sigma'} = \]

\[ = \frac{1}{(N!)^2} \sum_{\sigma, \sigma' \in \rho(\sigma)} P_{\sigma \sigma'} \]

Let \( \sigma^{(1)} = \sigma \sigma' \)

\[ \sigma' = \sigma \sigma^{(1)} \]

\[ \begin{align*}
\frac{\sigma'}{\sigma} &= \frac{\sigma^{(1)}}{\sigma} &= 1 \\
&= \frac{1}{(N!)^2} \sum_{\sigma \in \rho(\sigma)} \left( \sum_{\sigma'' \in \rho(\sigma)} P_{\sigma''} \right) =
\end{align*} \]
\[ \frac{1}{N} \sum_{6} s \cdot s = \frac{N'}{N!} s = s \]

\[ s^2 = s \]

Also not

\[ s^+ = \frac{1}{n} \sum_{6 \in \bar{G}(\nu)} p_6^+ = \]

\[ \frac{1}{n} \sum_{6 \in \bar{G}(\nu)} p_6^+ \cdot \]

\[ s^+ = s' \]

\[ \frac{1}{N} \left( \sum_{6'' \in \bar{G}(\nu)} p_{6''} \right) = \]

\[ \frac{1}{N} \left( \sum_{6'' \in \bar{G}(\nu)} p_{6''} \right) = s \]

\[ s = s^+ \]

\[ \text{Similarly,} \]

\[ A^2 = \frac{1}{(N!)^2} \sum_{6} (-) \quad p_{6} p_{6'} = \]

\[ A^2 = \frac{1}{(N!)^2} \sum_{6} (-) \quad p_{6} p_{6'} \]

\[ \text{Let} \quad G'' = G G' \quad (-) = (-) \]

\[ \text{Let} \quad G'' = G G' \quad (-) = (-) \]
\[
\frac{1}{(N!)^2} \sum_{S} \sum_{s} (-1)^{i} P_{6}^{s}
\]

\[
= \frac{1}{N!} \sum_{a} A = \frac{N!}{N!} A
\]

Similarly, \( A = A^{\dagger} \) uses the same argument used for \( S \) with \( 16'1 = 16 \)

since they both have the same # of transpositions.

Finally, not

\[
A S = \frac{1}{(N!)^2} \sum_{S} (-1)^{i} P_{6} P_{6}^{s} = \frac{1}{(N!)^2} \sum_{S} (-1)^{i} P_{6}^{s}
\]

\[
= \frac{1}{(N!)^2} \sum_{S} (-1)^{i} P_{6}^{s} \quad \text{let } 6'' = 66'
\]

\[
= \frac{1}{N!} \sum_{s} (-1)^{s} S
\]
but \( \sum (-1)^{N/2} = 0 \) because there are \( \frac{N!}{2} \) even and \( \frac{N!}{2} \) odd permutations.

So S and A are orthogonal projection operators.

If \( H \) is a Hamiltonian for a system of identical particles,

\[
\left[ S H \right] = \left[ A H \right] = 0
\]

the subspaces do not mix — it has symmetric and antisymmetric eigenstates, but only one is physically relevant depending on the spin of the particles.
Normalize

\[ |\Psi\rangle \rightarrow A|\Psi\rangle \text{ or } S|\Psi\rangle \]

If \[ |\Psi\rangle \] is not symmetric or antisymmetric

\[ |\hat{\Psi}\rangle = \frac{S|\Psi\rangle}{\langle \Phi S^+ | S |\Psi\rangle} = \frac{S|\Psi\rangle}{\langle \Phi S^+ | S |\Psi\rangle} = \frac{S|\Psi\rangle}{\langle \Phi S^+ | S |\Psi\rangle} \]

Similarly

\[ |\hat{\Phi}\rangle = \frac{A|\Psi\rangle}{\langle \Phi A^+ | A |\Psi\rangle} \]

Also note

\[ \langle \Phi | H | S^+ |\Psi\rangle = \]
\[ \langle \Phi | H | S^2 |\Psi\rangle = \]
\[ \langle \Phi | S | H | S |\Psi\rangle = \]
\[ \langle \Phi | S^+ | H | S |\Psi\rangle \]

Similarly,

\[ \langle \Phi | H | A |\Psi\rangle = \]
\[ \langle \Phi | A^+ | H | A |\Psi\rangle \]
which means that it is sufficient to symmetrize or antisymmetrize only the initial or final states (although they must be normalized.)

**Occupation # representation**

1. Start with an orthonormal basis of single particle states
   \[ \sum_{n=1}^{\infty} | \phi_n \rangle \]
   \[ \langle \phi_n | \phi_m \rangle = \delta_{nm} \]

2. Products on N such states are a basis for the N particle Hilbert space
\[ |\Psi\rangle = \sum (n, n) |\Phi_n\rangle |\Phi_m\rangle \]

3. when the particles are identical, all we know (up to sign) is how many particles are in state 1, state 2, ...

4. introduce a new representation called the occupation number representation

\[ |m_1, m_2, \ldots, m_n\rangle\]

- \(m_1\) identical particles in state \(\Phi_1\)
- \(m_2\) identical particles in state \(\Phi_2\)
- 

;
many identical particles
in state |p_n>

1 ≤ n ≤ ∞

This representation can be used to describe systems with any number of identical particles.

Note normally all but a finite number of the m_i are 0

$$N = \sum_{k=1}^{\infty} M_k$$

For Fermions, M_k = 0 or 1
For Bosons, M_k can be any non-negative integer.
In order to use this representation, we define

\[ |10\rangle = |000\rangle \Rightarrow \]

the state consisting of no particles. We also define

\[ \langle 010 \rangle = 1 \]

Next we define non-Hermitian operators:

\[ a_n^+ \text{ for bosons,} \]

\[ b_n^+ \text{ for fermions.} \]

\[ a_n^+ \text{ adds 1 particle in state } |a_n\rangle \]

\[ a_n \text{ removes 1 particle in state } |a_n\rangle \]
We require

$$[a_n^+ a_m] = \delta_{nm}$$
$$\sum b_\alpha b_\alpha^+ = \delta_{nm}$$

$$[a_n^+ a_m] = [a_n a_m]^+ = 0$$
$$\sum b_\alpha b_\alpha^+ = \sum b_\alpha^+ b_\alpha = 0$$

For Fermions,

$$b_n^+ b_m = - b_m b_n$$

so interchanging identical particles changes the sign of the state as

$$b_n^+ b_n = - b_n b_n^+ = 0$$
so we cannot to 2
fermions in the same
single particle state

ordering convention

$$(a_1^+)^m_1(a_2^+)^m_2 \cdots |0\rangle \equiv
1 m_1, m_2 m_3 \cdots \rightarrow N$$

$$(b_1^+)^m_1(b_2^+)^m_2 \cdots |0\rangle
1 m_1, m_2 m_3 \cdots \rightarrow N$$

where $N$ is a normalization
constant that must
be determined

in order to determine
the normalization not
\[ N_m \equiv a_m^+ a_m \]

\[
[ N_m, a_m ] = a_m^+ a_m a_m - a_m a_m a_m^+ = [ a_m^+, a_m ] a_m
\]

\[
= -a_m
\]

\[
[ N_m, a_m^+ ] = a_m^+ a_m a_m^+ - a_m a_m^+ a_m = a_m^+ [ a_m a_m^+ ]
\]

\[
= a_m^+ a_m
\]

\[ N_m \equiv b_m^+ b_m \]

\[
[ N_m, b_m ] = b_m^+ b_m b_m + b_m b_m^+ b_m
\]

\[
= b_m^+ b_m b_m + b_m b_m^+ b_m
\]

\[
= \sum b_m^+ b_m f b_m
\]

\[
= b_m
\]

\[
[ N_m, b_m^+ ] = b_m^+ b_m b_m^+ + b_m^+ b_m b_m
\]

\[
= b_m^+ \sum b_m b_m^+ b_m^+
\]

\[
= b_m^+
\]
Consider

\[ \begin{align*}
N_\text{R}(a^+)^n 10^- &= \\
N_\text{R}a^+(a^+)^{n-1} 10^- \\
a^+(N+1)(a^+)^{n-1} 10^- \\
(a^+)^2(N+2)(a^+)^{n-2} 10^- &= \\
(a^+)^n \text{ or } 10^- \\
N_\text{R}(a^+)^n 10^- &= \text{ m } 1(a^+)^n 10^-
\end{align*} \]

can also be done by

\[ \begin{align*}
N_\text{R}(a^+)^n 10^- &= \text{ m } (a^+)^n 10^- \\
N_\text{R}(a^+)^n 10^- &= \\
(N_\text{R}a^+ - a^+N_\text{R} + a^+N_\text{R})(a^+)^n 10^- \\
a^+_\text{R} (1+n)(a^+)^n 10^- &= \\
(N+1)(a^+)^n 10^-
\end{align*} \]
similar to $b_n^+$

\[ b_n^+ \ket{0} = \ket{n} \]

\[ b_n^+ \ket{n} = 0 \]

Normalization

\[ a_{k}^+ \ket{m_k} = \ket{m_{k+1}} c \]

\[ c^2 = \bra{m_n} a_{k} a_{k}^+ \ket{m_k} \]

\[ = \bra{m_n} a_{k} a_{k}^+ - a_{k} a_{k}^+ a_{k} a_{k}^+ \ket{m_k} \]

\[ = \bra{m_n} (1 + N_n) \ket{m_k} \]

\[ = (m_{k+1}) \bra{m_n} \ket{m_k} \]

\[ = m_{k+1} \]

\[ c = \sqrt{m_{k+1}} \]

\[ a_{k}^+ \ket{m_k} = \ket{m_{k+1}} \sqrt{m_{k+1}} \]

\[ (a_{k}^+)^n \ket{0} = \sqrt{n!} \ket{n+k} \]
1 m₁ m₂ \ldots =
\prod \frac{(a_{n}^{+})_{m_{n}}}{\sqrt{m_{n}}}

we can do the same
for fermions - we will have
M_n = l a \rightarrow l
we get
\prod \frac{(b_{n}^{+})_{m_{n}}}{\sqrt{m_{n}}}

although in this case
\prod \frac{(b_{n}^{+})_{m_{n}}}{\sqrt{m_{n}}} = \frac{(b_{1}^{+})_{m_{1}}}{\sqrt{m_{1}}} \frac{(b_{2}^{+})_{m_{2}}}{\sqrt{m_{2}}} \ldots

where the order matters
\[ |m_1 \ldots > = \left( \frac{a_1}{\sqrt{m_1}} \right)^{m_1} \left( \frac{a_2}{\sqrt{m_2}} \right)^{m_2} \ldots | 0 > \]

\[ <m_1 m_2 ... | 0 > = \sum_{m_1} \frac{(a_1)^{m_1}}{\sqrt{m_1}} \frac{(a_2)^{m_2}}{\sqrt{m_2}} \ldots \]

\[ |m_1 > = \left( \frac{b_1^+}{\sqrt{m_1}} \right)^{m_1} \left( \frac{b_2^+}{\sqrt{m_2}} \right)^{m_2} \ldots | 0 > \]

\[ <m_1 m_2 ... | 0 > = \sum_{m_1} \frac{(b_1^+)^{m_1}}{\sqrt{m_1}} \frac{(b_2^+)^{m_2}}{\sqrt{m_2}} \ldots \]

**Open atom**

\[ | \text{body open atom} > = \sum \frac{a^+}{m_n} a_m <m101n> a_n \]

\[ <m101n> = \int \Phi_m^*(\vec{r}) \Phi_{101}^*(\vec{r}') \Phi_n(\vec{r}) \, d^3r \, d^3r' \]
Consider

\[ \langle 1_1, 2_1, 1_2 | \Omega | 2_3, 1_2 \rangle \]

\[ \sum_{m} \frac{1}{\sqrt{2}!} \frac{(a_2)^2}{\sqrt{m}} a_m^{+} \langle m | \Omega | n \rangle a_n \times \]

\[ \frac{(a_3^{+})^2}{\sqrt{2}!} \frac{a_2^{+}}{\sqrt{2}!} 1_0 \rangle \]

remark

\[ a_n^{+} (a_n) = m (a_n^{+})^{m-1} \]

\[ (a_n)^{m} a_n^{+} = m (a_n)^{m-1} \]

this matrix element

\[ \langle 0_1 \frac{a_2}{\sqrt{2}} a_1^{+} \ 2 \langle 21 \ | \Omega | 12 \rangle \frac{(a_3^{+})^2}{\sqrt{2}!} 1_0 \rangle \]

\[ \langle 0_1 \frac{(a_2)^2}{\sqrt{2}!} \langle 11 \ | \Omega | 12 \rangle \frac{(a_3^{+})^{1}}{\sqrt{2}!} 1_0 \rangle \]

\[ \langle 0_1 \frac{a_2}{\sqrt{2}} a_1^{+} \ 2 \langle 21 \ | \Omega | 13 \rangle \frac{2}{\sqrt{2}!} a_3^{+} a_1^{+} 1_0 \rangle \]

\[ \langle 0_1 \frac{(a_2)^{1}}{\sqrt{2}!} \langle 11 \ | \Omega | 13 \rangle \frac{2}{\sqrt{2}!} a_3^{+} a_1^{+} 1_0 \rangle \]
This vanishes – to get something non zero all states have to be the same except one initial and one final state

\[ <01a_3a_2a_1|g|a_n^+<n|a_{11m}>a_{11}a_{22}a_{33}> \]

\[ <11a_{11}> + <21a_{12}> + <31a_{13}> \]

which gives the sum of all 3 single particle matrix elements

\[ <01a_3^2a_2a_1|g|a_n^+<n|a_{11m}>a_{11}a_{22}a_{33}> \]

\[ = \frac{1}{2} \cdot 2 \cdot 2 \cdot <31a_{13}> + <11a_{11}> + <21a_{12}> \]

\[ = 2 <31a_{13}> + <11a_{11}> + <21a_{12}> \]
Next we define 2 body operators. First note

\[ |n_1 n_2 \rangle = \frac{1}{\sqrt{2}} ( |n_1 \rangle |n_2 \rangle + |n_2 \rangle |n_1 \rangle ) \]

for bosons

\[ |n_1 n_2 \rangle = \frac{1}{\sqrt{2}} ( |n_1 \rangle |n_2 \rangle - |n_2 \rangle |n_1 \rangle ) \]

for fermions

\[ \langle n_1 n_2 \mid 1V \mid n_1 n_2 \rangle = \]

\[ \frac{1}{2} ( \langle n_1 \downarrow n_2 \downarrow \rangle \pm \langle n_1 \downarrow n_2 \uparrow \rangle ) V ( \langle n_1 \uparrow n_2 \uparrow \rangle \pm \langle n_1 \uparrow n_2 \downarrow \rangle ) \]

We define

\[ \hat{V} = \frac{1}{2} \sum a_{n_1}^+ a_{n_2}^+ \langle n_1 n_2 \downarrow \mid 1V \mid m_1 m_2 \rangle a_{m_2} a_{m_1} \]
Consider the expectation value of this operator in a 2-particle state

$$\langle 0 | a_2 a_1^+ 1 a_1^+ a_2^+ 1^0 \rangle =$$

$$\frac{1}{2} \left( \langle 0 \left| a_2 a_1^+ a_{n2}^+ a_{n1}^+ \langle \eta_1 \eta_2 \rangle \right| \lambda \rho \right)$$

$$a_{m2} a_{m1} | a_1^+ a_2^+ 1^0 \rangle =$$

$$\frac{1}{2} \langle 1 2 1 1 1 2 \rangle +$$

$$\frac{1}{2} \langle 2 1 1 1 2 1 \rangle +$$

$$\frac{1}{2} \langle 1 2 1 1 1 2 \rangle +$$

$$\frac{1}{2} \langle 2 1 1 1 1 2 \rangle =$$

$$\langle 1 2 1 1 1 2 \rangle + \langle 1 2 1 1 1 2 \rangle$$

Since

$$\langle 1 2 1 1 1 2 \rangle = \langle 2 1 1 1 1 2 \rangle$$

$$\langle 1 2 1 1 1 2 \rangle = \langle 2 1 1 1 1 2 \rangle$$
If these were fermions
the \( <12 | 12 > \) and \( < 21 | 11 > \) terms would have a
sign