Lecture 4

1 parameter unitary groups

definition $U(U)$

1. $U(\lambda) = U(\lambda)^\dagger$
2. $U(0) = I$
3. $U(\lambda_1)U(\lambda_2) = U(\lambda_1 + \lambda_2)$

examples - rotations about the $\hat{n}$ axis $\lambda = \text{angle of rotation}$

**Theorem** $U(\lambda) = e^{-iG\lambda}$ where $G = G^\dagger$ and $G$ is independent of $\lambda$

$G$ is called the infinitesimal generator of $U(U)$

**Proof**

$I = U(\lambda_1 + \lambda_2)U^\dagger(\lambda_1 + \lambda_2)$

$0 = \frac{d}{d\lambda_1} U(\lambda_1)U(\lambda_2)U^\dagger(\lambda_2)U^\dagger(U_1) + U(\lambda_1)U(\lambda_2)U^\dagger(U_2) \frac{dU^\dagger}{d\lambda_1}(\lambda_1)$

$\frac{d}{d\lambda_1} U(\lambda_1 + \lambda_2), U^\dagger(\lambda_1 + \lambda_2) =$
\[- \frac{d}{d\lambda} \left( \theta(\lambda_1, \lambda_2) U^\dagger(\lambda_1, \lambda_2) \right) \]

Note also: \( \lambda = \lambda_1 + \lambda_2 \), \( \frac{d}{d\lambda} = \frac{d}{d\lambda_1} + \frac{d}{d\lambda_2} = 1 \frac{d}{d\lambda} \)

This means

\[
\frac{d}{d\lambda} U(\lambda) U^\dagger(\lambda) = \frac{d}{d\lambda_1} U(\lambda_1) U^\dagger(\lambda_2) U(\lambda_1) = \frac{d}{d\lambda_1} U(\lambda_1)
\]

This means

\[
F(\lambda) = \frac{d}{d\lambda} U(\lambda) U^\dagger(\lambda) = F(\lambda_1) = - F^\dagger(\lambda)
\]

\[
F(\lambda_1 + \lambda_2) = F(\lambda_1)
\]

which means \( F \) is independent of \( \lambda \)

Let \( F = -iG \) then

\[(F^\dagger) = (-iG)^\dagger = iG = -F = iG
\]

so \( G = G^\dagger \)

\[
\frac{d}{d\lambda} U^\dagger(\lambda) = -iG
\]

\[
\frac{d}{d\lambda} U(\lambda) = -iG U(\lambda)
\]
\[ \frac{d^2 u}{d\lambda^2} = -i \sigma \frac{du}{d\lambda} \]

\[ \frac{d^n u}{d\lambda^n} = (-i\sigma)^n u(\lambda) \]

\[ u(\lambda) = I + \sum_{n=1}^{\infty} \frac{(-i\sigma)^n}{n!} I = e^{-i\lambda I} \]

If \( u(\lambda) \) satisfies

\[ u(\lambda) H u^+(\lambda) = H \quad u(\lambda) H = H u(\lambda) \]

This means that the transformation does not change \( H \). If this is true, the \( \sigma \) is a conserved quantity.

\[ \frac{dH}{d\lambda} = 0 = \frac{du}{d\lambda} H u^+ + u H \frac{dU}{d\lambda} \]

\[ = \frac{d}{d\lambda} (e^{-i\lambda I} H (e^{i\sigma I} + e^{i\sigma H} \frac{d}{d\lambda} (e^{i\sigma I}) \]

\[ = e^{-i\lambda I} (-i\sigma H + iH\sigma) e^{i\lambda I} = 0 \]

set \( \lambda = 0 \)

\[ -i [\sigma H] = 0 \quad \sigma [\sigma, H] = 0 \]
The Heisenberg field equations give

\[ \frac{d\phi}{dt} = \frac{i}{\hbar} [\hat{H}, \phi] = 0 \]

which means that \( \phi \) is a conserved quantity.

\[ \hat{U} \hat{H} \hat{U}^\dagger = \hat{H} \rightarrow \frac{d\phi}{dt} = 0 \]

Conversely, if \( \frac{d\phi}{dt} = 0 \) \( [\hat{H}, \phi] = 0 \)

\[ \frac{d}{d\lambda} \phi = -i \lambda [\hat{E}, \phi] \]

\[ \frac{d^n}{d\lambda^n} \phi = (-i)^n \lambda^n [\hat{E}, \phi] \]

This means

\[ \phi = \hat{H} \]

\[ \hat{H} = \hat{U} \hat{H} \hat{U}^\dagger \leftrightarrow \frac{d\phi}{dt} = 0 \]

This gives the connection between symmetries and conservation laws.
examples

1. \[ \frac{dH}{dt} = i \hbar \left[ H, H \right] + \frac{\partial H}{\partial t} \]

If \( \frac{\partial H}{\partial t} = 0 \) then \( H \) is a conserved quantity (energy) and

\[ e^{-iHt/\hbar} = U(t) \]

satisfies

\[ U(t) H U^\dagger(t) = H \]

2. consider

\[ U(a) \langle x | \Phi \rangle = \langle x-a | \Phi \rangle \]

\[ \begin{array}{c}
\varnothing \quad \rightarrow \\
\begin{array}{c}
0 \\
a
\end{array}
\end{array} \]

This translates the origin to the right by \( a \)

(1) \[ U(a) \langle x | \Phi \rangle = \langle x-a | \Phi \rangle \quad U(0) = I \]

\[ \int \phi^\dagger(x-a) \phi(x-a) \, dx = \chi = x-a \]

\[ \int \phi^\dagger(y) \phi(y) \, dy \]

\[ \langle \phi_1 U(a) \Phi_1 | \Phi \rangle = \langle \phi_1 | \Phi \rangle \]

\( U(a)^\dagger = U(a)^{-1} \)
(5) $U(a_2)U(a_1)\langle x\mid 1 \rangle = \\
U(a_2)\langle x-a_1\mid 1 \rangle = \langle x-a_1-a_2\mid 1 \rangle = \\
U(a_1+a_2)\langle x\mid 1 \rangle$

$U(a)$ is a $1$ parameter unitary group

$$\frac{d}{da} \ U(a) \langle x\mid 1 \rangle = \frac{d}{da} \langle x+a\mid 1 \rangle$$

$$-i\hbar \langle x+a\mid 1 \rangle = \frac{d}{dx} \langle x\mid 1 \rangle$$

Set $a=0$

$$-i\hbar \langle x\mid 1 \rangle = \frac{d}{dx} \langle x\mid 1 \rangle$$

$$G = i \frac{d}{dx} \langle x\mid 1 \rangle = -\frac{i}{\hbar} p$$

where $p = \frac{\hbar}{i} \frac{d}{dx}$ is the momentum operator

This means $i\frac{p \cdot a}{\hbar}$

(1) $U(a) = e$

(2) $[U(a), H] = 0 \Rightarrow P$ is a conserved quantity
Note that $\hbar$ is needed to make $x \cdot p$ dimensionless - otherwise the series $1 - i \hbar a + \frac{1}{2} (-i \hbar)^2 a^2$ is a series of quantities with different dimensions.

**Galilean Relativity**

In classical mechanics there are certain coordinate systems where Newton's second law for isolated systems has the same form - these are called inertial coordinate systems.

For a free particle consider
\[
\tilde{x} \rightarrow x' = R \tilde{x} - \tilde{v} t - \tilde{a} \\
\tilde{t}' = \tilde{t} - \tilde{t}_0
\]

where $R$ is a $3 \times 3$ rotation matrix $R^T R = I \quad \text{det} R = 1$
The transformation on the last page is called a Galilean transformation.

Hence, Galilean transformation leave Newton's second law for a free particle unchanged.

\[ m \frac{d^2 x}{dt^2} = 0 \]

\[ m \frac{d^2 x'}{dt^1} = m \left( \ddot{x} - \ddot{y} \frac{d^2 t}{dt^1} \right) = m \ddot{x} \]

This means if \( m \frac{d^2 x}{dt^2} = 0 \) then \( m \frac{d^2 x'}{dt^1} = 0 \).

The principle of Galilean relativity asserts that Newton's second law has the same form (for isolated systems) in all inertial coordinate systems.
In this case — for an isolated particle system —

\[ m_i \frac{d^2 \tilde{x}_i}{dt^2} = \mathbf{F}_i \left( \tilde{x}_i, \tilde{x}_v, \tilde{V}_i, \tilde{V}_v, t \right) \]

**Galilean invariance** requires:

1. \( \mathbf{F}_i \) are independent of time
   \[ \frac{\partial \mathbf{F}_i}{\partial t} = 0 \]
2. \( \mathbf{F}_i \) only depend on coordinate differences
3. \( \mathbf{F}_i \) only depend on velocity differences
4. \( R \mathbf{F}_i \left( \tilde{x}_i, \ldots, \tilde{x}_v, \tilde{V}_i, \ldots, \tilde{V}_v \right) = \mathbf{F}_i \left( R \tilde{x}_i, \ldots, R \tilde{x}_v, R \tilde{V}_i, \ldots, R \tilde{V}_v \right) \)

Galilean invariance implies:

Invariance under time translation, space translation, rotations, and velocity shifts.

In quantum mechanics some of these invariances lead to conservation laws.
It is useful to express Galilean transformation using a 5x5 matrix representation

\[
\begin{pmatrix}
R & -\mathbf{v} & -\mathbf{a} \\
0 & 1 & -t_0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
t \\
1
\end{pmatrix}
= 
\begin{pmatrix}
R\mathbf{x} - \mathbf{v}t - \mathbf{a} \\
t - t_0 \\
1
\end{pmatrix}
\]

Properties:

Set \( R = I \), \( \mathbf{v} = 0 \), \( \mathbf{a} = 0 \), \( t = 0 \)

Then

\[
q = \begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0
\end{pmatrix}
= \text{Identity}
\]

\[
\begin{pmatrix}
R_2 & -V_2 - a_2 \\
0 & 1 - t_{02} \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
R_1 & -V_1 - a_1 \\
0 & 1 - t_{01} \\
0 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
R_2 R_1 & -R_2 V_1 - V_2 - R_2 a_1 + V_2 t_{01} - a_2 \\
0 & 1 - t_{01} - t_{02} \\
0 & 0 & 1
\end{pmatrix}
\]

This shows that the product of 2 Galilean transformations is a Galilean transformation.
Choose
\[ R_2 = \tilde{R}_1 = R_1^T \]
\[ V_2 = R_2 V_1 = R_1^T V_1 \]
\[ a_2 = -R_2 a_1 + V_2 t_1 = -R_1^T a_1 + (R_2 V_1 t_1) \]
\[ t_{02} = -t_1 \]

This defines the inverse of the original transformation.

To product of Galilean transformations are Galilean transformations.

Every Galilean transformation has an inverse.

The identity is a Galilean transformation.

Products of matrices are associative, so Galilean transformations are associative.

The Galilean transformations form a group.
It is not hard to show

\[ q_e(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - t & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad q_R(\theta) = \begin{pmatrix} R(\theta_0) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ q_{0,i}(a_i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & a_i \\ 0 & 1 & 0 \end{pmatrix} \quad q_{N}(v_i) = \begin{pmatrix} 1 & -v_i & 0 \\ v_i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

are all parameter subgroups of the Galilean group.

What does Galilean invariance mean in quantum mechanics?

Difference with classical mechanics.

In mechanics \( \mathbf{x}_i(t) \) is the coordinate of a particle at time \( t \) - it can be measured.

In quantum mechanics the Schrödinger equation gives \( |\psi_i(t)\rangle \) as \( \rho(t) \). These cannot be measured.
The measurable quantities in a quantum theory are:

1. Probabilities $p = |\langle \phi | 1^+ \rangle|^2$

2. Expectation values

$$\langle \psi | A | \psi \rangle = \sum_{\text{quantum eigenvalues} \lambda} \psi_{\lambda}^2 \text{ probability} \quad \lambda = A^+$$

3. Ensemble average

$$\text{Tr} (pA) = \sum_{\text{classical probabilities}} p_m \langle m | \psi | n \rangle^2 \text{ probability}$$

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Principle of Galilean relativity in a quantum theory:

Quantum measurements cannot be used to distinguish inertial coordinate systems for isolated systems.
Wigner's Theorem

The most general transformation that preserves all quantum probabilities are either unitary

\[ |\Psi\rangle \rightarrow |\Psi'\rangle = U |\Psi\rangle \]

\[ \langle \phi' | \psi' \rangle = \langle \phi | \psi \rangle \]

or antunitary

\[ |\Psi\rangle \rightarrow |\Psi'\rangle = A |\Psi\rangle \]

\[ \langle \phi' | \psi' \rangle = \langle \psi | \phi \rangle = \langle \phi | \psi \rangle^* \]

Both of these clearly leave the probabilities unchanged. Wigner showed that these are the only possibilities. I have attached a proof of Wigner's theorem from Gottfried's text on quantum mechanics - I do not plan on covering it in class.
remark

antunitary unitors are not linear

\[ \langle \mathcal{A} \psi | \mathcal{A} (\phi_1 + \alpha \phi_2) \rangle = \]
\[ \langle \phi_1 + \alpha \phi_2 | \psi \rangle = \langle \phi_1 | \psi \rangle + \alpha \langle \phi_2 | \psi \rangle = \]
\[ \langle \mathcal{A} \psi | \mathcal{A} \phi_1 \rangle + \alpha \langle \mathcal{A} \psi | \mathcal{A} \phi_2 \rangle \]

since \( \mathcal{A} \psi \) is arbitrary

\[ \mathcal{A} (\phi_1 + \alpha \phi_2) = \mathcal{A} \phi_1 + \alpha \mathcal{A} \phi_2 \]

operators satisfying the above are called antilinear operators.

If \( \mathbf{B} | \psi \rangle = \mathbf{b} | \psi \rangle \), \( \mathbf{b} = \mathbf{b}^\ast \)

\[ \mathbf{A} \mathbf{B} \mathbf{A}^\dagger | \psi \rangle = \mathbf{A} (\mathbf{b} | \psi \rangle) = \mathbf{A} \mathbf{b} | \psi \rangle = \]
\[ \mathbf{b}^\ast \mathbf{A} | \psi \rangle = \mathbf{b} H | \psi \rangle = \mathbf{b} | \psi \rangle \]

also

\[ \mathbf{U} \mathbf{B} \mathbf{U}^\dagger | \psi \rangle = \mathbf{U} \mathbf{b} | \psi \rangle = \mathbf{b} \mathbf{U} | \psi \rangle = \mathbf{b} | \psi \rangle \]
This shows that both unitary and antunitary operators leave expectation values unchanged (eigenvalues and probabilities do not change).

The transformations on wave functions do not affect classical probabilities.

* The most general transformations that leave all quantum observable unchanged are unitary or antunitary transforms.

* Unlike classical mechanics, these transformations change the sum of the Schrödinger equation.

Theorem: One parameter groups cannot be antunitary.

\[ \langle A^4 \Phi \rangle = \langle \Phi \rangle \]
\[ \langle A^2 A^2 \Phi \rangle = \langle A^4 \Phi \rangle = \langle \Phi \rangle \]

So the square of an antunitary transformation is unitary since

\[ U(U) = U(U_2)U(U_2) \]
It is not hard to show that any Galilean transformation is a product of 1 parameter groups:

\[
U(t)U(\tilde{a})U(\tilde{v})U(R)
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -\tilde{a} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -\tilde{v} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Each of these 1 parameter groups are unitary - the product of unitary operators are unitary.

\[
\tilde{g}(R,\tilde{v},\tilde{a},t)
\]

denote the Galilean transformation:

\[
\begin{pmatrix}
\frac{R - \tilde{v} \cdot \tilde{a}}{\sqrt{\tilde{c}}}, -\tilde{a} \\
0, 1 - t
\end{pmatrix}
\]

We expect

\[
U(g_2)U(g_1) = U(g_2g_1)e^{i\tilde{\Phi}(g_2,g_1)}
\]
This means taking 2 steps to get to \( x_2 \) or 1 step should give
\[
\begin{align*}
x & \rightarrow x_2 \\
x & \rightarrow x_1 \rightarrow x_2
\end{align*}
\]
the same result. Since quantum states are defined up to a phase we must allow for that possibility.

It is possible to redefine \( U(q) \)
\[
U(q) \rightarrow U'(q) = U(q)e^{i\xi(q)}
\]
then
\[
U'(q_2)U'(q_1) = U'(q_2, q_1)e^{-i\xi(q_2, q_1) + i\xi(q_1) + i\Phi(q_2, q_1)}
\]

In some cases it is possible to choose \( \xi(q) \) so the overall phase vanishes - then
\[
U(q_2)U(q_1) = U(q_2, q_1)
\]
gives a unitary representation of the Galilean group; but for some groups this is not possible.
\[ q \to q \quad q_v \]
\[ x \to x - vt \quad t \to t - t' \quad x - vt - a \to x - vt - a \]
\[ q, q_v, (a + vt) q(t) \quad \left( \begin{array}{c} x \\ t - t_v \end{array} \right) \to x - a - vt \quad x - V(t - t') \quad t \to t - t' \]
\[ = \quad x - vt - q \]

Both sequences of transformation give \( g(\mathbf{I}, \mathbf{v}, \mathbf{a}, t) \), but if we consider

\[ U(g(a)) U(g(v)) U(g(v)) | E \bar{P} > = \]
\[ U(g(v)) U(g(a + vt)) U(g(v)) | E \bar{P} > \]

where \( E, P \) are eigenstates of Energy and momentum

\[ U(a) | E \bar{P} > = e^{i \frac{\mathbf{p} \cdot \mathbf{a}}{\hbar}} | E \bar{P} > \]
\[ U(t) | E \bar{P} > = e^{-i \frac{\mathbf{E} t}{\hbar}} | E \bar{P} > \]

This gives the operator

\[ i \frac{\mathbf{p} \cdot \mathbf{a}}{\hbar} - i \frac{\mathbf{E} t}{\hbar} \]

\[ E \quad e \quad U(v) \quad | E \bar{P} > = \]
\[ U(g(v)) \quad e^{i \frac{\mathbf{p}}{\hbar} (a + vt) - i \frac{\mathbf{E} t}{\hbar}} \quad | E \bar{P} > = \]
\[ i \frac{\mathbf{p}}{\hbar} a \quad i \frac{\mathbf{E}}{\hbar} (-E + \mathbf{v} \bar{P}) t \]
\[ e \quad e \quad U(g(v)) \quad | E \bar{P} > \]

number
\[ H \left| u(v) \right\rangle = (\tilde{E} - \hat{V} \cdot \tilde{p}) \left| u(v) \right\rangle \]
\[ \tilde{p} \left| u(v) \right\rangle = \tilde{p} \left| u(v) \right\rangle \]

This is not what we expect classically.

\[ p \rightarrow \tilde{p} - mv \]
\[ \frac{p^2}{2m} - (\frac{p-mv}{2m})^2 = \frac{p^2}{2m} - \tilde{p} \cdot \tilde{v} + \frac{1}{2} mv^2 \]

\[ E \rightarrow \tilde{E} - \tilde{p} \cdot \tilde{v} + \frac{1}{2} mv^2 \]
\[ \tilde{p} \rightarrow \tilde{p} - m\tilde{v} \]

It is clear that these transformations do not agree. In addition, there is a new parameter \( m \) that does not appear in the Galilean transformation.

This shows that without the phase the states will not transform as expected.
This is a situation where the phase cannot be transformed away without giving the incorrect transformation properties.

Bengmamn showed

\[ U(q_2)U(q_1) = U(q,q_1)e^{i\delta(q_2,q_1)} \]

\[ \delta(q,q_1) = -\frac{m}{2}\left( \bar{a}_2 \cdot \bar{R}_2 \bar{v}_1 - \bar{v}_2 \bar{R}_1 \bar{a}_1 + m \bar{v}_2 \cdot \bar{R}_1 \bar{v}_1 \right) \]

Fixes the transformation property

\( m \) looks like a new generator - it is called a central charge

\( m \) because it is a number it commutes with all of the generators of the Galilean group (this is why it is called central)

The term charge is due to Noether's theorem with respect to generators of "conserved charges"
Let \( q_1 = q_{v_1} \)
\[ q_2 = q_{a(a)} q_t(t) \]
\[ q_3 = q_{a(a+v_1)} q_t(t) \]
\[ q_2 q_1 = q_1, q_3 = q_4 \]
\[ U(q_2)U(q_1) = U(q_4), \quad e^{i\mathcal{S}(21)} \]
\[ U(q_1)U(q_3) = U(q_4), \quad e^{i\mathcal{S}(13)} \]
\[ \mathcal{S}(21) = -\frac{m}{2\hbar} (\mathbf{q}, \mathbf{v}) \]
\[ \mathcal{S}(13) = \frac{m}{2\hbar} (\mathbf{v} \cdot (\mathbf{a} + \mathbf{v} t)) \]
\[ U(q_2)U(q_1) = U(q_1)U(q_3) \quad e^{-i\mathcal{S}(13) + i\mathcal{S}(21)} \]
\[ U(q_1)U(q_3) = U(q_1)U(q_3) \quad e^{-i\frac{m}{2\hbar} (v \cdot a + v^2 t + v \cdot \mathbf{a})} \]

\[ e^{\frac{\mathbf{P}^2}{2\hbar} - i\hbar t/a} \]
\[ U(v) |E \rangle = e^{i\frac{\mathbf{P}^2}{2\hbar} + i(E-\mathbf{v} \cdot \mathbf{P}) - i\frac{m}{2\hbar} (2\mathbf{v} \cdot a + v^2 t)} \]
\[ U(v) |E \rangle \quad e^{i\frac{(P-mv) \cdot a}{\hbar} - i \frac{E - P \cdot \mathbf{v} + \frac{m v^2}{2}}{\hbar} t/a} \]

which gives the expected transformation properties.
\[ \frac{d}{dt} \rightarrow U(\hat{\mathbf{r}}, \mathbf{0}) H U^\dagger(\hat{\mathbf{r}}, \mathbf{0}) = H \]

\[ \frac{d}{dt} \rightarrow \hat{p}_a - i \mathcal{H} t / \hbar - i \mathcal{H} t / \hbar \]

\[ \mathbf{e} \mathbf{e} \mathbf{e} = \mathbf{e} \]

\[ \begin{align*}
\frac{d}{dt} & \quad U(\mathbf{a}) H U^\dagger(\mathbf{a}) = H \\
\end{align*} \]

A Galilean invariant Hamiltonian is invariant under translations and rotations. This means energy, momentum, and angular momentum are conserved in a Galilean invariant quantum theory.

\[ U(\mathbf{v}) H U^\dagger(\mathbf{v}) = H - \hat{p}_a + \mathbf{v} \hat{p}_a \]

The generator of velocity boost does not commute with \( H \).
Wigner's theorem allows for the possibility of antiunitary operators.

Consider time reversal - this is not a continuous transformation so antiunitary transformations are not ruled out.

Since \( U(t) = e^{-iHt/n} \)

generates time reversal we expect \( TU(t)T^{-1} = e^{iHt/n} \)

Here we use \( T^{-1} \) to allow for the possibility of a unitary or antiunitary transformation

\[
TU(t)T^{-1} = \sum_n \frac{1}{n!} (iH)^n T(T(iH)^n)T^{-1}
\]

\[
= \sum_n \frac{1}{n!} T(iH)^n T(iH)^n T(iH)^n \cdots T
\]

Here we kept \( i \) with \( H \) because if \( T \) is antiunitary, \( T \) changes the sign of \( i \).
In order to satisfy
\[ T U(t) T^{-1} = U(-t) \]
we must have
\[ THT^{-1} = -H \quad T \text{ unitary}, \]
\[ THT^{-1} = H \quad T \text{ antiunitary}. \]

If \( H \) is bounded from below but not above - we expect that the time reversed \( H \) will have the same property - otherwise time reversal would be very unstable.

Thus for \( THT^{-1} = H \) we must have
\[ T = \text{antiunitary}. \]