Lecture 6

The Galilean principle of relativity

\[ g(\mathbf{R}, \mathbf{v}, \mathbf{a}, t) = \begin{pmatrix} \mathbf{R} & \mathbf{v} & \mathbf{a} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix} \]

Successive Galilean transformations

\[ \begin{pmatrix} \mathbf{R}_2 & -\mathbf{v}_2 & -\mathbf{a}_2 \\ 0 & 1 & -t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}_1 & -\mathbf{v}_1 & -\mathbf{a}_1 \\ 0 & 1 & -t_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_2 \mathbf{R}_1 & -\mathbf{R}_2 \mathbf{v}_1 \mathbf{v}_2 & -\mathbf{R}_2 \mathbf{a}_1 + \mathbf{v}_2 t_1 \mathbf{a}_2 \\ 0 & 1 & -t_2 -t_1 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ g(\mathbf{R}_2 \mathbf{R}_1, \mathbf{R}_2 \mathbf{v}_1 + \mathbf{v}_2, \mathbf{R}_2 \mathbf{a}_1 - \mathbf{v}_2 t_1 + \mathbf{a}_2, t_1 + t_2) \]

Consider \( q_2 \) and \( q_1 \)

\[ g(I, v_2, 0, 0) g (I, 0, a_1 + v_2 t_1, t_1) = q^q \]

\[ g(I, v_2, a_1 + v_2 t_1 - v_2 t_1, t_1) = g(I, v, a, t) \]

\[ q_3 q_2 \]

\[ g(I, 0, a_2, t_2) g(I, v, 0, 0) \]

\[ g(I, v, a_2, t_2) \]

This gives

\[ q_4 = q_2 q_1 = q_3 q_2 \]
without phases

\[ U(q_4) = U(q_2) U(q_1) = U(q_3) U(q_2) \]

with phases

\[ U(q_2) U(q_1) = U(q_4) e^{i \Phi(q_3 q_1)} \]
\[ U(q_3) U(q_1) = U(q_4) e^{i \Phi(q_4 q_1)} \]

this gives

\[ U(q_4) = U(q_2) U(q_1) e^{-i \Phi(q_3 q_1)} \]
\[ = U(q_3) U(q_2) e^{-i \Phi(q_4 q_1)} \]
\[ U(q_3) U(q_2) = U(q_1) U(q_4) e^{-i \Phi(q_4 q_1) + i \Phi(q_2 q_3)} \]

Next we show that without the phases this gives a result that is inconsistent with the classical expectation.

since

\[ g(T_0, a_0) g(T_0, a_0) = g(T_0, a_0) g(T_0, a_0) \]
\[ U(T_0, 0 \ 0 \ 0) = e^{i p \cdot a_0 / \hbar} \]
\[ U(T_0, 0 \ 0 \ 0) = e^{-i H t / \hbar} \]

in the absence of phase

\[ e^{i p \cdot a_0 / \hbar} e^{-i H t / \hbar} = e^{-i H t / \hbar} e^{i p \cdot a_0 / \hbar} = 1 \]
\[ \sum p, H t = 0 \]
This means that it is possible to find simultaneous eigenstates of $H, \tilde{\rho}$

\[
\begin{align*}
\tilde{\rho}, a_n - iht/n \\
\tilde{\rho}, n_t/n
\end{align*}
\]

Consider

\[
U(g(I, \tilde{v}, 0_0)) U(g(I, 0, \tilde{a} + \tilde{v}t, t)) \left< \tilde{\rho} \right>
\]

\[
U(g(I, \tilde{v}, 0_0)) \left< \tilde{\rho}, (\tilde{a} + \tilde{v}t)/n - iht/n \right>
\]

\[
\left< \tilde{\rho}, a_n/n - i(E - \tilde{v}n)t/n \right> U(g(I, \tilde{v}, 0_0)) \left< \tilde{\rho} \right>
\]

On the other hand

\[
\left< \tilde{\rho}, a_n/n - iht/n \right> U(g(I, \tilde{v}, 0_0)) \left< \tilde{\rho} \right>
\]

This shows that $U(g(I, \tilde{v}, 0_0)) \left< \tilde{\rho} \right>$ is an eigenstate of $H, \tilde{\rho}$

with eigenvalue

\[
E - \tilde{\rho} \cdot \tilde{v} \quad (H) \\
\tilde{\rho} \quad (\tilde{\rho})
\]
We expect
\[ p \rightarrow \bar{p} - m \bar{v} \]
\[ \frac{p^2}{2m} - (\bar{p} - m \bar{v})^2 = \frac{p^2}{2m} - \bar{p} \cdot \bar{v} + \frac{1}{2} m v^2 \]
\[ E \rightarrow E - \bar{p} \cdot \bar{v} + \frac{1}{2} m v^2 \]

We see that \( m \bar{v} \) and \( \frac{1}{2} m v^2 \) do not appear in the transformation; if we do not include a phase, we do not include a phase.

*Bargmann showed that the following choice of phase gives the correct galilean transformation properties.*

\[ x \text{ does not appear in the galilean Lie algebra.} \]

\[ U(R_2, v_2, a_2, t_2) U(R_1, v_1, a_1, t_1) = U(R_2 R_1, v_1 + v_1, R_2 R_1 - v_2, t_1 + t_1) \times e^{i \frac{1}{2} \hbar \left( \bar{a}_2 R_2 v_2 - \bar{v}_2 R_2 \bar{a}_2 + v_2 R_2 v_2 \right)} \]

We check that this gives the desired transformation properties.
applying \( \text{this to} \)

\[
\begin{align*}
U(q_3)U(q_2)|_{E,\mathbf{p}} &= \\
U(q_2)U(q_1)e^{-i\left(\phi(q_2,q_1) - \phi(q_3,q_2)\right)}|_{E,\mathbf{p}} \\
\phi(q_2,q_1) &= \frac{m}{2\hbar}(-\mathbf{\bar{v}} \cdot \mathbf{\bar{a}}) \\
\phi(q_3,q_1) &= \frac{m}{2\hbar}((\mathbf{a} + \mathbf{v}t) \cdot \mathbf{v}) \\
\phi(q_2,q_1) - \phi(q_3,q_1) &= \frac{m}{2\hbar}(-\mathbf{\bar{v}} \cdot \mathbf{\bar{a}} - \mathbf{\bar{a}} \cdot \mathbf{\bar{v}} - \mathbf{v}^2t)
\end{align*}
\]

If \( \text{we} \)

\[
\begin{align*}
\frac{i}{\hbar} (\mathbf{\bar{p}} \cdot \mathbf{\bar{a}}/\hbar - i(\bar{E} - \bar{p} \cdot \mathbf{\bar{v}})^{1/2} - \frac{m}{2}(-2\mathbf{\bar{v}} \cdot \mathbf{\bar{a}} - \mathbf{v}^2t)) \\
\mathbf{\bar{E}} &\quad \mathbf{\bar{p}} \end{align*}
\]

\[
\begin{align*}
\frac{i}{\hbar} (\mathbf{\bar{p}} - m\mathbf{\bar{v}}) \cdot \mathbf{\bar{a}}/\hbar - i(\mathbf{\bar{E}} + \frac{m(\mathbf{\bar{v}})^2}{2} - \mathbf{\bar{p}} \cdot \mathbf{\bar{v}})^{1/2} \\
\mathbf{\bar{E}} &\quad \mathbf{\bar{p}} \end{align*}
\]

\( \text{this gives} \)

\[
\begin{align*}
\mathbf{\bar{p}} \rightarrow \mathbf{\bar{p}}' &= \mathbf{\bar{p}} - m\mathbf{\bar{v}} \\
\mathbf{\bar{E}} \rightarrow \mathbf{\bar{E}}' &= \mathbf{\bar{E}} - \mathbf{\bar{p}} \cdot \mathbf{\bar{v}} + \frac{1}{2}m(\mathbf{\bar{v}})^2
\end{align*}
\]

\( \text{as expected} \).

The quantity \( m = \text{the inertial mass} \)

\( \text{is called the central charge}. \)
It is called central because it commuted with all of the Galilean generators.

It is called a change because in Lagrangian mechanics - transformations that leave the action invariant result in currents. The generators of the transformations are obtained by integrating the 0-component of the current over space - these are called changes - in analogy with current conservation.

Representations of Galilean group that are unitary up to phase are called projective representations.

The group can be extended to an 11 parameter group if the mass is considered as a generator.

Time reversal and antilinear transformations.
Recall the unitary time evolution operator is
\[ U(t) = e^{-i \frac{\text{H} t}{\hbar}} \]
For time reversal, we expect
\[ T U(t) T^{-1} = U(-t) \]
Here we use the inverse because at this point it is not clear if \( T \) should be unitary or antiunitary.

\[ T U(t) T^{-1} = T \sum \frac{1}{n!} (-i \frac{\text{H}}{\hbar})^n T^{-1} \]
\[ \sum \frac{1}{n!} T (-i \frac{\text{H}}{\hbar}) T^{-1} T (-i \frac{\text{H}}{\hbar}) T^{-1} \cdots T T (-i \frac{\text{H}}{\hbar}) T^{-1} \]

For this to equal \( U(-t) \) we must have
\[ T (-i \frac{\text{H}}{\hbar} t) T^{-1} = -i \frac{\text{H}}{\hbar} t \]

There are two ways to achieve this:
\[ T \text{H} T^{-1} = -\text{H} \quad T \text{ unitary} \]
\[ T \text{H} T^{-1} = \text{H} \quad T \text{ antiunitary} \]
Remarks

\( \ddot{x} \) position should not depend on
the direction of time

\( T \ddot{x} T^{-1} = \ddot{x} \)

\( \dot{p} = \frac{\hbar}{i} \frac{\partial}{\partial x} \) momentum should change

\( T p T^{-1} = p \) for \( T \) unitary, \( T \ddot{x} T^{-1} = \ddot{x} \)

\( T p T^{-1} = -p \) for \( T \) antunitary

\( T \frac{\hbar^2}{2m} T^{-1} = \frac{T p T^{-1} T p T^{-1}}{2m} = \frac{(-\ddot{p})(-\ddot{p})}{2m} = \frac{\dot{p}^2}{2m} \)

\( T K E T^{-1} = K E \)

If we consider \( H = \frac{\dot{p}^2}{2m} + V(x) \)

then

\( T H T^{-1} = H \)

More generally, if \( T H T^{-1} = -H \) then

if \( H \) is unbounded from above then

\( -H \) is unbounded from below —

this would result in an unstable system

so we choose

\( T H T^{-1} = H \)

\( T \) antunitary
summary
\[ T \tilde{T}^{-1} = \tilde{x} \]
\[ T \tilde{\bar{p}} T^{-1} = -\bar{p} \]
\[ T \tilde{H} T^{-1} = \tilde{H} \]
\[ T \text{ antilinear} \]

angular momentum
\[ T \tilde{L} T^{-1} = \tilde{T} (\tilde{x} \times \tilde{p}) T^{-1} = (T \tilde{x} T^{-1}) (T \tilde{p} T^{-1}) = \]
\[ \tilde{x} \times (\tilde{p}) = -\tilde{L} \]

If we want \( \tilde{J} = \tilde{L} + \tilde{S} \to \tilde{J}' = \pm \tilde{J} \) then we also choose \( \tilde{J} \) and \( \tilde{\bar{p}} \) to transform like \( \tilde{L} \),

\[ T \tilde{J} T^{-1} = \tilde{J} \]
\[ T \tilde{\bar{p}} T^{-1} = \tilde{\bar{p}} \]

in general,
\[ J_z T \mid j , \mu \rangle = T (T \tilde{J}_z T) \mid j , \mu \rangle \]
\[ = -T \tilde{J}_z \mid j , \mu \rangle \]
\[ = -\mu \mid j , \mu \rangle \]

This means
\[ T \mid j , \mu \rangle = \mid j , -\mu \rangle e^{-i\phi(\mu)} \]

next consider
\[ T \tilde{J}_z T^{-1} = \tilde{T} (J_x \pm iJ_y) T^{-1} = -J_x \pm iJ_y \]
\[ = - (J_x \mp iJ_y) \]
\[ = - J_\pm \]
\[ TJ^\pm |\mu\rangle = TJ^\pm T^{\dagger} T^{\dagger} |\mu\rangle \]
\[ = -J^\pm |\mp -\mu\rangle e^{i\Phi(\mu)} \]
\[ \sqrt{(g \mp \mu) |\mu + 1\rangle |\mu \pm 1\rangle} \]
\[ \sqrt{(g \mp \mu)(\mp \mu + 1)} |\mp \mu \pm 1\rangle e^{i\Phi(\mu + 1)} \]

Comparing both sides:
\[ i\Phi(\mu) \]
\[ |\mp -\mu \pm 1\rangle e = -|\mp -\mu \pm 1\rangle e \]

It follows that \( \Phi(\mu + 1) - \Phi(\mu) = -1 \)

It is still possible to include a \( \mu \)-dependent phase:
\[ i\Phi(\mu) = \frac{\delta \mu}{2} \]

This has the advantage that \( \frac{\delta \mu}{2} \) is always an integer, even for spin \( \frac{1}{2} \):
\[ \sqrt{g - \mu} = |\sqrt{g} - \mu\rangle e^{i\phi(\mu)} \]

Consequences:

1. Spin \( \frac{1}{2} \)

\[ T^2 |\mu\rangle = T |\sqrt{g} - \mu\rangle e^{-i\phi(\mu)} = 1^2 |\mu\rangle e^{2i\phi(\mu)} = |\mu\rangle e^{i\phi(\mu)} \]
For half integral spins
\[ T^2 |\frac{1}{2}\mu\rangle = -|\frac{1}{2}\mu\rangle \]

Note if \( THT^{-1} = H \) (\( H \) is invariant under time reversal)

\[ H |\frac{1}{2}\mu\rangle = T T^{-1} H |\frac{1}{2}\mu\rangle \]
\[ = T H |\frac{1}{2}\mu\rangle \]
\[ = E |\frac{1}{2}\mu\rangle \]

\( |\frac{1}{2}\mu\rangle \) is also an eigenstate of \( H \) with the same energy.

If \( H \) is also rotationally invariant

\[ T |\frac{1}{2}\mu\rangle = |\frac{1}{2}\mu\rangle (-)^{J_{\mu}} \]

If the system consists of an odd number of spin \( \frac{1}{2} \) half particles, the spin necessarily changes under time reversal. This means that the states are degenerate. This is called the Kramers degeneracy.
Time reversal is an example of a discrete symmetry.

Another discrete symmetry is space reflection.

For space reflection, we expect

\[ P^2 = I \]
\[ P \bar{x} P^\dagger = -\bar{x} \quad \text{(position)} \]
\[ P \vec{p} P^\dagger = -\vec{p} \quad \text{(momentum)} \]

Note

\[ P (\bar{x} \times \vec{p}) P^\dagger = P \bar{x} P^\dagger \times P \vec{p} P^\dagger = -\bar{x} \times (-\vec{p}) = \bar{x} \times \vec{p} \]

\[ P \bar{E} P^\dagger = \bar{E} \]

We also assume that this holds for any anisotropic momentum.

\[ P \bar{s} P^\dagger = \bar{s} \]

**Def:** A pseudo vector is an operator that transforms like a vector under rotations, but remains unchanged under space reflection.

**Def:** A pseudo scalar is an operator that transforms like a scalar under rotations, but changes sign under space reflection.
Pseudoscalars can be constructed by taking scalar products of a vector and pseudovector since reflecting twice does nothing:

\[ p^2 = 0 \]
\[ p^2 - I = 0 = (p - I)(p + I) \]

This means that \( p \) has eigenvalues \( \pm 1 \):

\[ 1_+ = (p + I)1_+ = 1_+ \]
\[ p1_+ = 1_+ \]
\[ 1_- = (p - I)1_- = -1_- \]
\[ p1_- = -1_- \]

This means \( p = p^+ \), \( p^+ = p^\prime = p \).

Spherical Harmonics:

\( r^2 y^e_m(\theta \phi) \) is a homogeneous polynomial in the components of \( \mathbf{x} \) of degree \( e \).

Next week HW - show this is true for \( y^0_0, r y^1_m, r^2 y^2_m \).

It follows that:

\[ p1_{e m} = (-)^e 1_{e m} \]
note that \( u \) does not change because \( \psi \) is a pseudo vector

\[ J_z P \mid \psi u \rangle = P J_z \mid \psi u \rangle = \mu P \mid \psi u \rangle \]

selectim rules

(1) If \( [H, P] = 0 \) then we can label the eigenstates of \( H \) by their parity

\[ \frac{1}{\sqrt{2}} (1 + P) \mid 1E \rangle = \mid 1E^+ \rangle \]

\[ \frac{1}{\sqrt{2}} (1 - P) \mid 1E \rangle = \mid 1E^- \rangle \]

(it is possible that one of these states vanishes)

(2) \( P O_+ P = O_+ \) positive parity operator

\( P O_- P = O_- \) negative parity operator

\[ \langle E' + 1 O_+ \mid 1E^- \rangle = 0 \]

\[ \langle E' + 1 O_- \mid 1E^+ \rangle = 0 \]

selectim rules -

\[ \langle E' + 1 O \mid 1E^+ \rangle \neq 0 \quad O \text{ is positive parity} \]

\[ \langle E' + 1 O \mid 1E^- \rangle \neq 0 \quad O \text{ is negative parity} \]
where this is useful
\[
\langle \ell m_1 \sigma \ell' m_1' \rangle = 0 \quad \ell - \ell' \text{ odd}
\]
\[
\langle \ell m_1 0 \ell' m_1' \rangle = 0 \quad \ell - \ell' \text{ even}
\]

example \( Q = e \hat{r} \) electric dipole moment
is an odd parity operator.

Another type of symmetry occurs when the potential is periodic

\[
U(\vec{a}) H U(\vec{a})^\dagger = H
\]
\[
U(\vec{a}) \langle x \mid 14 \rangle = \langle x - \vec{a} \mid 14 \rangle
\]
\( \vec{a} \) is a fixed vector

\[ H \mid 14 \rangle = E \mid 14 \rangle \]
\[ H U(\vec{a}) \mid 14 \rangle = U(\vec{a}) H \mid 14 \rangle = E U(\vec{a}) \mid 14 \rangle \]

so \( U(\vec{a}) \mid 14 \rangle \) is also an eigenstate
with the same energy
\( U(\vec{a}) \mid 14 \rangle \) is also an eigenstate
with the same energy.
\[
\text{define} \quad |n\rangle = u(a)^n |E\rangle
\]

\[
\text{define} \quad |0\rangle = \sum_{n=-\infty}^\infty e^{-i\theta} |n\rangle
\]

\[
u(0)|0\rangle = \sum_{n=-\infty}^\infty e^{-i(n+1)\theta} |n\rangle + \sum_{n=-\infty}^\infty e^{-i(n+1)\theta} |n+1\rangle
\]

\[
\text{let} \quad n' = n+1
\]

\[
= \sum_{n'=-\infty}^\infty e^{-i\theta} e^{i\theta} |n'\rangle
\]

\[
= e^{-i\theta} |0\rangle
\]

\[
u(0)|0\rangle = e^{-i\theta} |0\rangle
\]

\[
|1\rangle = E |0\rangle
\]

Consider the general structure of

\[
\langle x | 1 \rangle = \langle x | u(0) | 0 \rangle = \langle x - a | 0 \rangle = e^{-i\theta} \langle x | 0 \rangle
\]

It follows that

\[
\langle x - na | 0 \rangle = e^{-i\theta} \langle x | 0 \rangle
\]
Define $f(x)$ by
\[ <x|\alpha> = e^{i k x} f(x) \quad k = \Theta/a \]

It follows that
\[ <x+\alpha|\alpha> = e^{i k (x+\alpha)} f(x+\alpha) \]
\[ = e^{i k x} e^{i \theta} f(x+\alpha) = e^{i \theta} <x|\alpha> \]
\[ <x|\alpha> = e^{i k x} f(x) \]

so
\[ e^{i \theta} e^{-i k x} = e^{i \theta} e^{-i k x} f(x) \]

canceling $e^{i \theta} e^{-i k x}$ from both sides of the equation gives
\[ <x|\alpha> = e^{i k x} f(x) \]
\[ k = \Theta/a \quad f(x+\alpha) = f(x) \]

The statement that it is possible to find energy eigenstates $<x|\alpha>$ of the above form with $f(x)$ periodic is called the Bloch theorem.
Consider

\[ H \langle x | \phi \rangle = E \langle x | \phi \rangle \]

\[ H = \frac{p^2}{2m} + V(x) \quad V(x) = V(x + a) \]

\[ \langle x | \phi \rangle = e^{ikx} \phi(x) \quad k = \frac{E}{\hbar} \]

\[ p e^{ikx} = e^{ikx}(\hbar k + p) \]

\[ p^2 e^{ikx} = p e^{ikx}(\hbar k + p) \]

\[ = e^{ikx}(\hbar k + p)^2 \]

\[ \left( \frac{p^2}{2m} + V(x) \right) e^{ikx} \phi(x) = E e^{ikx} \phi(x) = \]

\[ e^{ikx} \left( \frac{(p + \hbar k)^2}{2m} + V(x) \right) \phi(x) = E e^{ikx} \phi(x) \]

Canceling \( e^{ikx} \)

\[ \left( \frac{(p + \hbar k)^2}{2m} + V(x) \right) \phi(x) = E \phi(x) \]

Boundary condition \( \phi(x) = \phi(x + a) \)

This gives a differential equation for the periodic part of the wave function.