29:5742 Homework Assignment #3
Due 2/3/23

1. Consider the spherical harmonics $Y_0^0(\hat{x})$, $r^2 Y_1^1(\hat{x})$, $r^2 Y_2^2(\hat{x})$ where $r^2 = x^2 + y^2 + z^2$, $z = r \cos(\theta)$, $x = r \sin(\theta) \cos(\phi)$ and $y = r \sin(\theta) \sin(\phi)$. Express these spherical harmonics in terms of Cartesian coordinates and verify for these spherical harmonics that

$$Y_m^l(-\hat{x}) = (-1)^l Y_m^l(\hat{x})$$

2. Consider eigenstates of a Hydrogen atom $|n, l, m\mu\rangle$ where $n$ is the principal quantum number, $l$ is the total orbital angular momentum, $m$ is the orbital magnetic quantum number and $\mu$ is the spin projection.

Show the results of time reversal, space reflection, and combined space and time reflection on these states.

3. Consider the matrix element of the operator $L \cdot S$ in the states of problem 2

$$\langle n, l, m\mu | (L \cdot S) | n', l', m'\mu' \rangle$$

which combination of the quantum numbers can lead to non-zero matrix elements?

4. Protons and neutrons have very similar properties. It has been suggested that they are two states of a single particle called a nucleon with a quantum number called isospin. A nucleon has isospin $I = 1/2$ and the proton is the isospin up state in the z direction and the neutron is the isospin down state in the z direction. The Pauli matrices and the identity can be used to make general operators that depend on isospin.

Let $H$ be an interaction between two nucleons. Assume that it is invariant under isospin rotations. For two nucleons there are 4 possible isospin states. $H$ commutes with $I \cdot I$ and $\hat{z} \cdot I$. Discuss the structure and particle content of the four simultaneous eigenstates of the these three operators.

5. Let $X = ctI + x \cdot \sigma$ be a $2 \times 2$ matrix where $\sigma$ represents the three Pauli matrices considered as a vector. Let $A$ be a $2 \times 2$ matrix with $det(A) = 1$. Consider the transformation

$$X' = AXA^\dagger$$

Show that $(cr)^2 := (ct)^2 - x \cdot x = (ct')^2 - x' \cdot x'$. Show that successive transformation of this type preserve $(cr)^2$. This group is called $SL(2, C)$ - it is closely related to the group of Lorentz transformations, although $A$ and $-A$ give the same transformation.

6. The parity operator changes the sign of all component of $x$. Looking in a mirror only changes the sign of the direction normal to the mirror. How are these two type of transformations related?
Spherical Harmonics

\[ Y_0^0 = \frac{1}{\sqrt{4\pi}} \]

\[ r \ Y_1^0 = \sqrt{\frac{3}{4\pi}} \ r \ \cos \phi = \sqrt{\frac{3}{4\pi}} \ z \]

\[ r \ Y_1^\pm = \mp \sqrt{\frac{3}{8\pi}} \ r \ \sin \phi (\cos \phi \pm i \sin \phi) = \mp \sqrt{\frac{3}{8\pi}} \ (x \pm iy) \]

\[ r^2 \ Y_2^0 = \sqrt{\frac{5}{4\pi}} \ (\frac{3}{2} r \cos \theta - \frac{1}{2} r^2) = \sqrt{\frac{5}{4\pi}} \ \frac{1}{2} \ (2z^2 - x^2 - y^2) \]

\[ r^2 \ Y_2^\pm = \mp \sqrt{\frac{15}{8\pi}} \ (r \sin \theta \cos \phi)(\cos \phi \mp i \sin \phi) = \mp \sqrt{\frac{15}{8\pi}} \ z (x \pm iy) \]

\[ r^2 \ Y_2^{\pm 2} = \frac{1}{4} \sqrt{\frac{15}{8\pi}} \ (\sin \theta \ e^{i \phi})^2 = \frac{1}{4} \sqrt{\frac{15}{8\pi}} \ (x \pm iy)^2 \]

which is consistent with \( Y_{\pm}^n (\hat{r}) = e^\pm i \phi \ Y_{n}^m (\hat{\phi}) \)

\( \langle n' \ell' m' \mu' | L \ S | n \ell m \mu \rangle = (-1)^{\ell - \ell'} \langle n' \ell' m' - \mu' \mu \rangle \]

\( P \langle n \ell m \mu | L \ S | n' \ell' m' \mu' \rangle = (-1)^{\ell + \ell'} \langle n \ell m \mu | L \ S | n' \ell' m' \mu' \rangle \]

\( <n \ell m \mu | L \ S | n' \ell' m' \mu' > \)

\( L \cdot S = L_x S_x + L_y S_y + L_z S_z \)

Parity \( \Rightarrow \ L = L_z \) \( [L_z, \vec{L} \cdot \vec{S}] = 0 \)

\( m = m' \) \( \mu = \mu' \)

\( m' = m + 1 \) \( \mu' = \mu - 1 \)

\( m' = m - 1 \) \( \mu' = \mu + 1 \)

are all allowed
4) $| \frac{1}{2} \rangle$: 2 protons in isospin triplet

$\frac{1}{\sqrt{2}} (| \frac{1}{2} \rangle + | -\frac{1}{2} \rangle)$: 1 proton 1 neutron in an isospin triplet

$| -\frac{1}{2} \rangle$: 2 neutrons in an isospin triplet

$\frac{1}{\sqrt{2}} (| \frac{1}{2} \rangle - | -\frac{1}{2} \rangle)$: 1 proton and one neutron is an isospin singlet

The symmetry suggests that to a good approximation, the potential between any 2 triplet states will be approximately the same; the singlet could be different

5) $X = \begin{pmatrix} ct^2 + z & x - iy \\ x + iy & ct - z \end{pmatrix}$

$\det X = (ct + z)(ct - z) - (x + iy)(x - iy)$

$= c^2t^2 - z^2 - x^2 - y^2$

$= c^2t^2 - \vec{x} \cdot \vec{x}$
For $x' = AXA^t$

$$\det x' = (ct')^2 - (x':x') =$$

$$\det (AXA^t) = \det A \det X \det A^t = \det X$$

$$(ct')^2 - x:x$$

This is always preserved

$$x'' = BAXA^tB^t$$

$$\det x'' = \det B \det A \det X \det A^t \det B^t =$$

$$= \det x$$

6. $(x,y,z) \rightarrow (x,y,-z)$

Rotation about the $z$ axis by $180^\circ$ transform

$x \rightarrow -x$  $y \rightarrow -y$  $z \rightarrow z$

$\hat{p} = R(180^\circ) \cdot \Pi(z \rightarrow -z)$