Field theory in a wavelet basis

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What are wavelets?

- They are orthonormal basis functions that are used in data compression algorithms.
- JPEG digital images are tables of expansion coefficients in a wavelet basis.
- FBI fingerprint files are stored as expansion coefficients in a wavelet basis.

Wavelets in field theory?

- Complete set of local observables: orthonormal basis functions with compact support.
- Natural long and short wavelength cutoffs.
- Basis functions are fixed points of a linear renormalization group transform.
- Have more smoothness than block spins.
- Natural separation of scales.
- Exact multi-scale representation of field operators.

Operators

$$\underbrace{(Df)(x) = \sqrt{2}f(2x)}_{\textbf{scale change}} \qquad \underbrace{(Tf)(x) = f(x-1)}_{\textbf{translation}}.$$

Scaling equation

$$\phi(x) = D(\sum_{l=0}^{2K-1} h_l T^l \phi(x)) \qquad \int \phi(x)dx = 1$$

$\phi(x)$ **:= Scaling function**

Renormalization group transformation

$$f'(x) = \underbrace{D\,(\underbrace{\sum_{l=0}^{2K-1} h_l T^l f(x)(x))}_{\textbf{block average}}}_{\textbf{rescaling}}$$

$\phi(x)$ **is a fixed point of the Renormalization group transformation!**

h_n constant coefficients satisfying

$$\sum_{n=0}^{2K-1} h_n = \sqrt{2}$$

$$\sum_{n=0}^{2K-1} h_n h_{n-2m} = \delta_{m0}$$

$$g_n := (-l)^n h_{2K-1-n} \qquad \sum_{n=0}^{2K-1} n^m g_n = 0 \qquad m < K$$

Equations fix h_n up to reflection, $h_n \rightarrow h'_n = h_{2K-1-n}$

Daubechies' scaling coefficients, $K = 1, 2, 3$

h_l	**K=1**	**K=2**	**K=3**
h_0	$1/\sqrt{2}$	$(1+\sqrt{3})/4\sqrt{2}$	$(1+\sqrt{10}+\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_1	$1/\sqrt{2}$	$(3+\sqrt{3})/4\sqrt{2}$	$(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_2	0	$(3-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_3	0	$(1-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_4	0	0	$(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_5	0	0	$(1+\sqrt{10}-\sqrt{5+2\sqrt{10}})/16\sqrt{2}$

Properties of scaling function $\phi(x)$

1. **Reality**

$$\phi(x) = \phi^*(x)$$

2. **Partition of unity**

$$1 = \sum_{n=-\infty}^{\infty} \phi(x-n) = \sum_{n=-\infty}^{\infty} (T^n\phi)(x)$$

3. **Compact support**

$$\textbf{support}[\phi(x)] = [0, 2K-1]$$

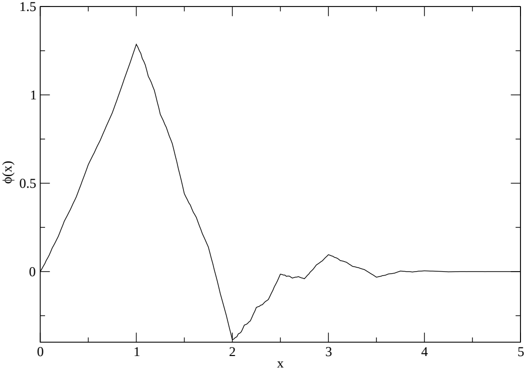
5. **Differentiability (K>2)**

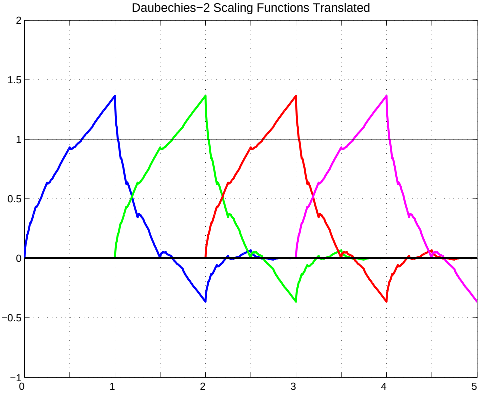
$$\frac{d\phi(x)}{dx} \quad \textbf{exists} \quad C^1(\mathbb{R}) \quad \textbf{for} \quad K \geq 3$$

6. **Orthonormality**

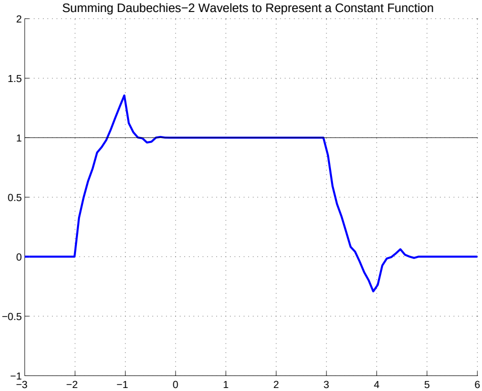
$$(T^m\phi, T^n\phi) = \delta_{mn}.$$

Daubechies' K=3 scaling function




Daubechies−2 Scaling Functions Translated
2
1.5
1
0.5
0
−0.5
−1
0
1
2
3
4
5

Partition of unity



Scaling properties

$$\phi_n^k(x) := (D^k T^n \phi)(x) = \sqrt{2^k}\phi(2^k(x - n/2^k))$$

Resolution $1/2^k$ subspace

$$\mathcal{V}_k := \{f(x) | f(x) = \sum_{n=-\infty}^{\infty} c_n \phi_n^k(x) \qquad \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty\}$$

Subspaces of different resolution related by

$$L^2(\mathbb{R}) \supset \cdots \supset \mathcal{V}_{k+1} \supset \mathcal{V}_k \supset \mathcal{V}_{k-1} \supset \cdots$$

Properties of $\phi_n^k(x)$

1. **Reality:**

$$\phi_n^k(x) = \phi_n^{k*}(x)$$

2. **Partition of unity:**

$$\frac{1}{\sqrt{2^k}} \sum_{n=-\infty}^{\infty} \phi_n^k(x) = \sum_{n=-\infty}^{\infty} \phi(2^k x - n) = 1$$

3. **Compact support:**

$$\textbf{support}[\phi_n^k(x)] = [\frac{n}{2^k}, \frac{n+2K-1}{2^k}]$$

4. **Differentiability (continuous for $k \geq 3$):**

$$\frac{d\phi_n^k(x)}{dx} = 2^k D^k T^n \frac{d\phi}{dx}$$

$$\frac{d}{dx} D = 2D \frac{d}{dx} \qquad \frac{d}{dx} T = T \frac{d}{dx}$$

5. **Orthonormality:**

$$(\phi_m^k, \phi_n^k) = \delta_{mn}$$

6. **Approximation:**

$$\lim_{k\to\infty} \mathcal{V}_k = L^2(\mathbb{R})$$

7. **Normalization (scale fixing):**

$$\int \phi_n^k(x)dx = \frac{1}{\sqrt{2^k}}$$

Multi-scale decomposition of $L^2(\mathbb{R})$

$$m > n \quad \Rightarrow \quad \mathcal{V}_m \supset \mathcal{V}_n$$

$$L^2(\mathbb{R}) \supset \cdots \supset \mathcal{V}_{n+1} \supset \mathcal{V}_n \supset \mathcal{V}_{n-1} \supset \cdots \supset \emptyset$$

$$\mathcal{V}_{n+1} = \mathcal{V}_n \oplus \mathcal{W}_n$$

$$\Downarrow$$

$$\mathcal{V}_n = \mathcal{W}_{n-1} \oplus \mathcal{W}_{n-2} \oplus \cdots \oplus \mathcal{W}_{n-m} \oplus \mathcal{V}_{n-m}$$

Theorem: $\lim_{n\to\infty} \mathcal{V}_n = L^2(\mathbb{R})$

$$L^2(\mathbb{R}) = \bigoplus_{n=-\infty}^{\infty} \mathcal{W}_n = \mathcal{V}_m \oplus \left(\bigoplus_{n=m}^{\infty} \mathcal{W}_n \right)$$

Wavelets

$\mathcal{W}_n$ **are wavelet spaces**

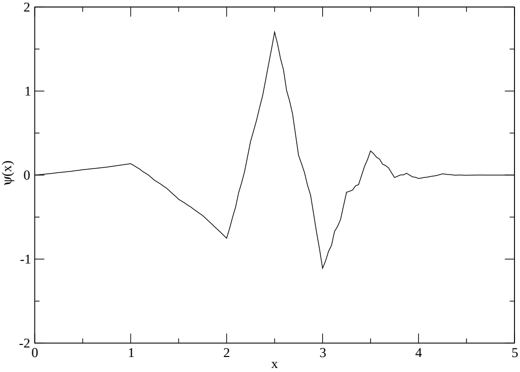
$$\psi(x) = D\left(\sum_{l=0}^{2K-1} (-)^l h_{2K-l-1} T^l \phi(x)\right)$$

$\psi(x)$ **is called the "Mother" wavelet**

$$\psi_l^n(x) := D^n T^l \psi(x)$$

$\{\psi_l^n\}_l$ **orthonormal basis for** $\mathcal{W}_n$

Daubechies' $K = 3$ mother wavelet



$$\textbf{support } [\psi(x)] = \textbf{support } [\phi(x)]$$

h_l **are determined up to space reflection by the requirements**

$$(\psi, x^n) = 0, \quad n = 0, \cdots, K-1 \qquad (\phi, T^m \phi) = \delta_{m0}$$

$$\Downarrow$$

$$\sum_m m^n (-)^m h_{l-m} = 0 \qquad \sum_{l=0}^{2K-1} h_{l-2m} h_l = \delta_{m0}$$

$$\sum_{l=1}^{2K-1} h_l = \sqrt{2}$$

Change of scale (expression coarse-scale functions in terms of fine-scale scaling functions):

$$\mathcal{V}_k = \mathcal{W}_{k-1} \oplus \mathcal{V}_{k-1}$$

$$\phi_n^{k-1}(x) = \underbrace{\sum_{l=0}^{2K-1} h_l \phi_{2n+l}^k(x)}_{\textbf{wavelet block-spin average}}$$

$$\psi_n^{k-1}(x) = \underbrace{\sum_{l=0}^{2K-1} g_l \phi_{2n+l}^k(x)}_{\textbf{lost high-frequency information}}$$

$$g_l = (-)^l h_{2K-1-l}$$

Inverse relations

Reconstruct fine resolution from coarse resolution plus wavelets

$$\phi_n^k = \sum_m h_{n-2m}\phi_m^{k-1} + \sum_m g_{n-2m}\psi_m^{k-1}$$

Wavelet localized fields (scale k)

$$\mathbf{\Phi}(x,t), \qquad \mathbf{\Pi}(x,t)$$

$$\mathbf{\Phi}^k(n,t) := \int \phi_n^k(x)\mathbf{\Phi}(x,t)dx$$

$$\tilde{\mathbf{\Phi}}^k(n,t) := \int \psi_n^k(x)\mathbf{\Phi}(x,t)dx$$

$$\mathbf{\Pi}^k(n,t) := \int \phi_n^k(x)\mathbf{\Pi}(x,t)dx$$

$$\tilde{\mathbf{\Pi}}^k(n,t) := \int \psi_n^k(x)\mathbf{\Pi}(x,t)dx$$

- **Two-kinds of fields: scaling function fields, $\mathbf{\Phi}^k(n,t)$, and wavelet fields, $\tilde{\mathbf{\Phi}}^k(n,t)$.**

- **Wavelet fields encode fine scale physics ($\frac{1}{2^k}$).**

- k **gives a short-distance cutoff.**

- **Limits $-N \leq n \leq N$ give volume cutoff.**

Commutation relations (fixed scale k): follow from

$$[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\delta(\mathbf{x}-\mathbf{y})$$

$$\Downarrow$$

$$[\mathbf{\Phi}^k(m,t),\mathbf{\Phi}^k(n,t)] = 0 \qquad [\mathbf{\Pi}^k(m,t),\mathbf{\Pi}^k(n,t)] = 0$$

$$[\mathbf{\Phi}^k(m,t),\mathbf{\Pi}^k(n,t)] = i\delta_{mn}$$

$$[\tilde{\mathbf{\Phi}}^k(m,t),\tilde{\mathbf{\Phi}}^k(n,t)] = 0 \qquad [\tilde{\mathbf{\Pi}}^k(m,t),\tilde{\mathbf{\Pi}}^k(n,t)] = 0$$

$$[\tilde{\mathbf{\Phi}}^k(m,t),\tilde{\mathbf{\Pi}}^k(n,t)] = i\delta_{mn}$$

$$[\tilde{\mathbf{\Phi}}^k(m,t),\mathbf{\Phi}^k(n,t)] = 0 \qquad [\tilde{\mathbf{\Pi}}^k(m,t),\mathbf{\Pi}^k(n,t)] = 0$$

$$[\tilde{\mathbf{\Phi}}^k(m,t),\mathbf{\Pi}^k(n,t)] = 0$$

Resolution $1/2^k$ field operators

$$\mathbf{\Phi}^k(x,t) = \sum_{n=-\infty}^{\infty} \mathbf{\Phi}^k(n,t)\phi_n^k(x)$$

$$\tilde{\mathbf{\Phi}}^k(x,t) = \sum_{n=-\infty}^{\infty} \tilde{\mathbf{\Phi}}^k(n,t)\psi_n^k(x)$$

$$\mathbf{\Pi}^k(x,t) = \sum_{n=-\infty}^{\infty} \mathbf{\Pi}^k(n,t)\phi_n^k(x)$$

$$\tilde{\mathbf{\Pi}}^k(x,t) = \sum_{n=-\infty}^{\infty} \tilde{\mathbf{\Pi}}^k(n,t)\psi_n^k(x)$$

Multiscale representation

Expansion of the exact field in terms of finite resolution parts

(k arbitrary but fixed coarse scale)

$$\mathbf{\Phi}(x,t) = \mathbf{\Phi}^k(x,t) + \sum_{m=k+1}^{\infty} \tilde{\mathbf{\Phi}}^m(x,t)$$

$$\mathbf{\Pi}(x,t) = \mathbf{\Pi}^k(x,t) + \sum_{m=k+1}^{\infty} \tilde{\mathbf{\Pi}}^m(x,t)$$

Space-time creation and annihilation operators

$$\mathbf{a}^k(n,t) := \frac{1}{\sqrt{2}}(\mathbf{\Phi}^k(n,t) + i\mathbf{\Pi}^k(n,t))$$

$$(\mathbf{a}^k(n,t))^\dagger := \frac{1}{\sqrt{2}}(\mathbf{\Phi}^k(n,t) - i\mathbf{\Pi}^k(n,t))$$

$$\tilde{\mathbf{a}}^k(n,t) := \frac{1}{\sqrt{2}}(\tilde{\mathbf{\Phi}}^k(n,t) + i\tilde{\mathbf{\Pi}}^k(n,t))$$

$$(\tilde{\mathbf{a}}^k(n,t))^\dagger := \frac{1}{\sqrt{2}}(\tilde{\mathbf{\Phi}}^k(n,t) - i\tilde{\mathbf{\Pi}}^k(n,t))$$

$$[\mathbf{a}^k(m,t), (\mathbf{a}^k(n,t))^\dagger] = \delta_{mn}$$

$$[\tilde{\mathbf{a}}^s(m,t), (\tilde{\mathbf{a}}^t(m,t))^\dagger] = \delta_{st}\delta_{mn}$$

All other commutators vanish

Can we use this algebra to describe an interacting field theory

Algebra can be given in terms of fields restricted to a light-front $^{x} = 0$.

States on the algebra are represented by discrete sequences.

$$e^{-Sa^{\dagger}}He^{Sa^{\dagger}}|0\rangle?$$

Interactions are almost local - ISU code?.

- **Only the very large** m **wavelet fields** $\mathbf{\Psi}^m$**are sensitive to short distance physics.**

- $\mathbf{\Phi}^k(x,t)$ **and** $\mathbf{\Pi}^k(x,t)$ **are Fock space operators - not operator-valued distributions!**

- $[\mathbf{\Phi}^k(x,t), \mathbf{\Phi}^k(y,t)] = 0$ **for** $|x-y| > \frac{2K-1}{2^k} \cdots$

- **Truncating the** n **sum gives a volume cutoff.**

- **Truncating the** m **sum gives a UV cutoff.**

Questions

- Field equations, Hamiltonians, require local operator products?
- Field equations, Hamiltonians, require derivatives of operators?
- How do we approximately realize the Poincare Lie algebra?
- How do we implement local gauge invariance?
- Reflection positivity?
- Momentum space?

Local products of fields

$$\mathbf{\Phi}^k(x,t)\mathbf{\Phi}^k(x,t) = \sum_{mn} \mathbf{\Phi}^k(m,t)\mathbf{\Phi}^k(n,t)\phi^k_m(x)\phi^k_n(x) \approx$$

$$\sum_{mnk} \mathbf{\Phi}^k(m,t)\mathbf{\Phi}^k(n,t)\Gamma^k_{mnl}\phi^k_l(x)$$

where

$$\Gamma^k_{mnl} = \int \phi^k_m(x,t)\phi^k_n(x,t)\phi^k_l(x,t)dx$$

can be computed exactly using scaling equation!

vanishes for $|n-m|$**,** $|n-l|$**,** $|m-l| > 2K-1$

Products become slightly non-local

Spatial derivatives of fields

$$\frac{d\mathbf{\Phi}^k}{dx}(x,t) = \sum_{mn} \mathbf{\Phi}^k(m,t)\frac{d\phi_m^k}{dx}(x) \approx$$

$$\sum_{mn} \mathbf{\Phi}^k(m,t) D_{mn}^k \phi_n^k(x)$$

where

$$D_{mn}^k = \int \frac{d\phi_m^k}{dx}(x,t)\phi_n^k(x,t)dx$$

can be computed exactly using scaling equation!

vanishes for $|n-m| > 2K-1$

Properties of $\Gamma^k_{n_1,\cdots,n_m}$ and $D^k_{m,n}$

$$\Gamma^k_{n_1,\cdots,n_m} = \Gamma^k_{n_{\sigma(1)},\cdots,n_{\sigma(m)}}$$

for any permutation of m objects σ.

$$\sum_{n_1} \Gamma^k_{n_1,\cdots,n_m} = \sqrt{2^k}\Gamma^k_{n_2,\cdots,n_m}$$

$$\Gamma^k_{n_1,\cdots,n_m} = 2^{k(\frac{n-2}{2})}\Gamma^0_{n_1,\cdots,n_m}$$

$$D^k_{m,n} = -D^k_{n,m}$$

$$\sum_n D_{mn} = -\sum_n D_{nm} = 0$$

$$D^k_{m,n} = 2^k D^0_{m,n}.$$

Exact local products

$$\begin{aligned}
\mathbf{\Phi}(x,t)\mathbf{\Phi}(x,t) &= \sum \mathbf{\Phi}^{k}(n,t)\mathbf{\Phi}^{k}(m,t)(\Gamma^{kkk}_{nmr}\phi^{k}_{r}(x) + \sum \Gamma^{kk\dot{l}}_{nmr}\psi^{l}_{r}(x)) \\
&+ \sum \tilde{\mathbf{\Phi}}^{l}(n,t)\mathbf{\Phi}^{k}(m,t)(\sum \Gamma^{\dot{l}kk}_{nmr}\phi^{k}_{r}(x) + \sum \Gamma^{\dot{l}k\dot{l}'}_{nmr}\psi^{l'}_{r}(x)) \\
&+ \sum \mathbf{\Phi}^{k}(n,t)\tilde{\mathbf{\Phi}}^{l}(m,t)(\sum \Gamma^{k\dot{l}k}_{nmr}\phi^{k}_{r}(x) + \sum \Gamma^{k\dot{l}\dot{l}'}_{nmr}\psi^{l'}_{r}(x)) \\
&+ \sum \tilde{\mathbf{\Phi}}^{l}(n,t)\tilde{\mathbf{\Phi}}^{s}(m,t)(\sum \Gamma^{\dot{l}\dot{s}k}_{nmr}\phi^{k}_{r}(x) + \sum \Gamma^{\dot{l}\dot{s}\dot{l}'}_{nmr}\psi^{l'}_{r}(x))
\end{aligned}$$

Exact local derivatives

$$\frac{d\mathbf{\Phi}(x,t)}{dx} = \sum \mathbf{\Phi}^{k}(n)(D^{kk}_{nm}\phi^{k}_{m}(x) + \sum D^{k\dot{l}}_{nm}\psi^{l}_{m}(x))$$

Local products constructed out of infinite sets of non-local products

$$\Gamma^{k_1 \cdots k_m}_{n_1 \cdots n_m} = \int \phi^{k_1}_{n_1}(x) \phi^{k_2}_{n_2} \cdots (x) \phi^{k_n}_{n_m}(x) dx$$

$$\Gamma^{\dot{k}_1 \cdots k_m}_{n_1 \cdots n_m} = \int \psi^{k_1}_{n_1}(x) \cdots \phi^{k_m}_{n_m}(x) dx$$

$$\vdots$$

$$D^{\dot{k}_1 k_2}_{n_1 n_2} = \int \frac{d\psi^{k_1}_{n_1}}{dx}(x) \cdots \phi^{k_m}_{n_m}(x) dx$$

$$\vdots$$

Can all be computed exactly using the scaling equation

Renormalization group properties of fields

$$\mathbf{\Phi}^k(n,t) = \sum_m h_{n-2m}\mathbf{\Phi}^{k-1}(m,t) + \sum_m g_{n-2m}\tilde{\mathbf{\Phi}}^{k-1}(m,t) =$$

$$D\mathbf{\Phi}^k(n,t) := (\phi_n^k, D\mathbf{\Phi}) = (D^{-1}\phi_n^k, \mathbf{\Phi}) = \Phi^{k-1}(n,t)$$

$$\mathbf{\Phi}^k(n,t) = D(\sum_m h_{n-2m}\mathbf{\Phi}^k(m,t) + \sum_m g_{n-2m}\tilde{\mathbf{\Phi}}^k(m,t))$$

Symmetries

$$[O^a(x), O^b(y)] = i\delta(x-y) f^{abc} O^c(y)$$

$$1 = (2^{-k/2} \sum_n \phi_n^k(x))(2^{-k/2} \sum_m \phi_m^k(y))$$

$$1 = (2^{-k/2} \sum_n \phi_n^k(x))$$

$$O_n^{ak} := 2^{k/2} \int O^a(x)\phi_n^k(x)dx$$

$$[\sum_n O_n^{ak}, \sum_m O_m^{bk}] = if^{abc} \sum_l O_l^{ck}$$

$$O^a \to \sum_n O_n^{ak}$$

Gives local generators - the symmetry is broken when products of discrete files are use to construct O_n^{ak}.

Gauge transformations

$$\mathbf{\Phi}_n^k \to \mathbf{\Phi}_n'^k = V_n \mathbf{\Phi}_n^k$$

Must preserve both operator products and derivatives

$$\sum_{mnl} \int \mathbf{\Psi}_n^{k\prime\dagger} \mathbf{\Psi}_m^{k\prime} \Gamma_{nml}^k \phi_l^k(x) dx =$$

$$\sum_{mn} \mathbf{\Psi}_n^{k\prime\dagger} \mathbf{\Psi}_m^{k\prime} 2^{-k} \delta_{nm} 2^k =$$

$$\sum_{n} \mathbf{\Psi}_n^{k\prime\dagger} \mathbf{\Psi}_n^{k\prime} =$$

$$\sum_{n} \mathbf{\Psi}_n^{k\dagger} V^\dagger(n) V(n) \mathbf{\Psi}_n^k =$$

$$\sum_{n} \mathbf{\Psi}_n^{k\dagger} \mathbf{\Psi}_n^k$$

$$-i\int \mathbf{\Psi}^{\dagger}(x)\frac{d}{dx}\mathbf{\Psi}(x)dx$$

The Discrete version of this is

$$-i\sum \mathbf{\Psi}_n^{\dagger k}\mathbf{\Psi}_m^k D_{ml}^k \Gamma_{nlr}^k \int \phi_r^k(x)$$

Using $(D_{mn}^k = -D_{nm}^k)$ **this becomes**

$$-i\sum \mathbf{\Psi}_n^{\dagger k}\mathbf{\Psi}_m^k D_{mn}^k = \sum_{nm} \mathbf{\Psi}_n^{\dagger k}(iD_{nm}^k)\mathbf{\Psi}_m^k$$

Under Gauge transformations

$$\sum_{nm} \mathbf{\Psi}_n^{\dagger k}(iD_{nm}^k)\mathbf{\Psi}_m^k D_{mn}^k \rightarrow$$

$$\sum_{nm} \mathbf{\Psi}_n^{\dagger k\prime}(iD_{nm}^k)\mathbf{\Psi}_m^{k\prime} =$$

$$\sum_{nm} \mathbf{\Psi}_n^{\dagger k} V^\dagger(n)(iD_{nm}^k)V(m)\mathbf{\Psi}_m^k$$

$$\sum_{nm} \mathbf{\Psi}_n^{\dagger k} (iD_{nm}^k) \mathbf{\Psi}_m^k$$

$$\Downarrow$$

Introduce a vector potential

$$\sum_{nm} \mathbf{\Psi}_n^{\dagger k} (iD_{nm}^k - e\mathbf{A}_{mn}^k) \mathbf{\Psi}_m^k$$

Gauge invariance requires

$$\sum_{nm} \mathbf{\Psi}_n^{\dagger k\prime} (iD_{nm}^k - e\mathbf{A}_{mn}^{k\prime}) \mathbf{\Psi}_m^{k\prime} =$$

$$\sum_{nm} \mathbf{\Psi}_n^{\dagger k} V^\dagger(n) (iD_{nm}^k - e\mathbf{A}_{nm}^k) V(m) \mathbf{\Psi}_m^k$$

Gauge covariant derivative

$$(iD^{k}_{nm} - e\mathbf{A}^{k\prime}_{mn})$$

Gauge transformation of $\mathbf{A}^{k}_{mn}$

$$\mathbf{A}^{k\prime}_{nm} = iD^{k}_{nm} - iV^{\dagger}(n)D^{k}_{nm}V(m) + V^{\dagger}(n)\mathbf{A}^{k\prime}_{nm}V^{\dagger}(m)$$

Outlook

- Different discrete approach to field theory.
- Renormalization group and block spin transformations natural.
- Has structural features to treat Gauge theories and symmetries.
- Replaces all operator distributions with almost local operators with support on all scales.
- Infinities due infinite sums.
- Space-time creation and annihilation operators.

Discussion: Useful applications ?

Thanks James!